

# Weak convergence of the localized disturbance flow to the coalescing Brownian flow

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## Abstract

We define a new state-space for the coalescing Brownian flow on the circle. This space is a complete separable metric space of maps on the circle with a certain weak flow property and having continuous time-dependence. A larger state-space, allowing jumps in time, is also introduced, and equipped with a Skorokhod-type metric. We prove that the coalescing Brownian flow is the weak limit in this larger space of a family of discrete-time flows generated by small localized disturbances of the circle. A local version of this result is also obtained, in which the weak limit law is that of the coalescing Brownian flow on the line. Our set-up is well adapted to time-reversal and our weak limit result provides a new proof of time-reversibility of the coalescing Brownian flow. We also identify a martingale associated with the coalescing Brownian flow on the circle and use this to make a direct calculation of the Laplace transform of the time to complete coalescence. We finally explore the relationship between our formulation of the coalescing Brownian flow and the Brownian web.

## 1 Introduction

This paper is a contribution to the theory of stochastic flows in one dimension. Our main result, Theorem 6.1, establishes the weak convergence of a certain class of stochastic flows, which we call disturbance flows, to the coalescing Brownian flow. This is motivated by a surprising connection with a model of Hastings and Levitov [9] for planar aggregation, which is worked out in our companion paper [15]. In this model, the flow of harmonic measure on the cluster boundary is a disturbance flow, and our convergence theorem then shows that that the random structure of fingers in the Hasting–Levitov cluster is well described in the small-particle limit by the coalescing Brownian flow.

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Disturbance flows, which we define in the next section, are discrete in time, whilst acting, possibly discontinuously, but preserving order, on the circle. In seeking a suitable metric space on which to consider weak convergence of such flows, we were led to introduce some new spaces, which we call the continuous weak flow space and the cadlag weak flow space. These spaces have a number of convenient properties, which we prove, and indeed provide a good framework for weak convergence of one-dimensional flows.

The coalescing Brownian flow is, loosely speaking, a family of one-dimensional Brownian motions, one for each space-time starting point, which evolve independently up to collision and coalesce thereafter. The possibility to identify a precise mathematical object corresponding to this idea was shown by Arratia in 1979 in his PhD thesis [1], where the model was considered as a limit object for coalescing random walks on the integers. Subsequent work was done by many people including Harris [8], who was interested in general coalescing stochastic flows, and Piterbarg [16], who showed that Arratia's flow arises as a weak limit of rescaled isotropic stochastic flows. Further properties were developed by Tóth and Werner [18] in 1998, who used the flow to construct an object which they called the 'continuous true self-repelling motion'. Arratia's flow forms an important example in Tsirelson's general theory of non-classical stochastic flows [19]. Fontes, Isopi, Newman and Ravishankar [6, 7] introduced the name 'Brownian web' in 2004, to describe a number of new formulations of Arratia's flow, and gave further characterization and convergence results. The paper [6] characterizes the Brownian web as a random element of a space of compact collections of paths with specified starting points. This characterization is used by Ferrari, Fontes and Wu [5] to show that the Brownian web arises as a limit of two-dimensional Poisson trees. More recently, Sun and Swart [17] and Newman, Ravishankar and Schertzer [13] introduced a new object, the 'Brownian net', which is constructed from pairs of Brownian webs. This work is closely related to the dynamical Brownian web, proposed by Howitt and Warren [10], which builds on the theory of stochastic flows of kernels due to Le Jan and Raimond [12]. In this paper we follow most closely the viewpoint of Tóth and Werner but lay greater stress on the almost sure flow-type properties. Disturbance flows and the continuous and cadlag weak flow spaces do not seem to have been studied before. The relation between our work and the alternative and well-established framework from [6] is discussed in the Appendix.

This paper is organised as follows. In Section 2 we introduce the disturbance flows, a natural class of spatially homogeneous, order preserving, discrete-time random flows on the circle, and show a weak convergence result for the time-evolutions of finitely many points, in the limit as the disturbances become small and well-localized. In Section 3 we define the continuous weak flow space and show that it provides a canonical space for the coalescing Brownian flow. Section 4 is a short digression on the distribution of the time taken for the coalescing Brownian flow on the circle to coalesce completely. In Section 5, the larger cadlag weak flow space, of Skorokhod type, is introduced. This is a complete separable metric space suitable for the formulation of weak convergence of stochastic flows which are

not necessarily continuous in time or space. The convergence of the disturbance flow to the coalescing Brownian flow, is shown in Section 6. Finally, in Section 7 we take advantage of the approximation by disturbance flows to give a new proof of the time-reversibility of the coalescing Brownian flow. An early version of some parts of the present paper, along with its companion paper [15], appeared in [14]. We are grateful to Tom Ellis for helpful suggestions during the writing of this paper.

## 2 The disturbance flow on the circle

We introduce a class of random flows on the circle, whose distributions are invariant under rotations of the circle and under which each point on the circle performs a random walk. The flow maps are in general not continuous on the circle but have a non-crossing property. In a certain asymptotic regime, the motion of the flow from a countable family of starting points is shown to converge weakly to a family of coalescing Brownian motions.

We specify a particular flow by the choice of a non-decreasing, right-continuous function  $f^+ : \mathbb{R} \rightarrow \mathbb{R}$  with the following *degree 1* property<sup>4</sup>

$$f^+(x+n) = f^+(x) + n, \quad x \in \mathbb{R}, \quad n \in \mathbb{Z}. \quad (1)$$

Denote the set of such functions by  $\mathcal{R}$  and write  $\mathcal{L}$  for the analogous set of left-continuous functions. Each  $f^+ \in \mathcal{R}$  has a left-continuous modification  $f^- \in \mathcal{L}$ , given by  $f^-(x) = \lim_{y \uparrow x} f^+(y)$ . Write  $\mathcal{D}$  for the set of all pairs  $f = \{f^-, f^+\}$ . When  $f^+$  is continuous, we also write  $f = f^+$  and, generally, we write  $f$  in place of  $f^\pm$  in expressions where the choice of left or right-continuous modification makes no difference to the value. The sets  $\mathcal{R}$  and  $\mathcal{L}$  are closed under composition, but  $\mathcal{D}$  is not. In fact, if  $f_1, f_2 \in \mathcal{D}$ , then  $f_2^- \circ f_1^-$  is the left-continuous modification of  $f_2^+ \circ f_1^+$  if and only if  $f_1$  sends no interval of positive length to a point of discontinuity of  $f_2$ . We say in this case that  $f_2 \circ f_1 \in \mathcal{D}$ , denoting by  $f_2 \circ f_1$  the pair  $\{f_2^- \circ f_1^-, f_2^+ \circ f_1^+\}$ . Write  $\tilde{f}^\pm$  for the periodic functions  $\tilde{f}^\pm(x) = f^\pm(x) - x$ . Define  $\text{id}(x) = x$  and set

$$\mathcal{D}^* = \left\{ f \in \mathcal{D} \setminus \{\text{id}\} : \int_0^1 \tilde{f}(x) dx = 0 \right\}.$$

We assume throughout that our basic map  $f \in \mathcal{D}^*$ .

Let us suppose we are given a sequence  $(\Theta_n : n \in \mathbb{Z})$  of independent random variables, all distributed uniformly on  $(0, 1]$ . For  $f \in \mathcal{D}$  and  $\theta \in (0, 1]$ , define  $f_\theta(x) = f(x - \theta) + \theta$ . Then define, for  $m, n \in \mathbb{Z}$  with  $m < n$ ,

$$\Phi_{n,m}^\pm = f_{\Theta_n}^\pm \circ \cdots \circ f_{\Theta_{m+1}}^\pm.$$

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<sup>4</sup>These functions can be considered as liftings of maps from the circle  $\mathbb{R}/\mathbb{Z}$  to itself having an order-preserving property. In the limiting regime which we consider, the circle map is a perturbation of the identity map and our basic map  $f^+$  is the unique lifting which is close to the identity map on  $\mathbb{R}$ .

Set  $\Phi_{n,n} = \text{id}$  for all  $n \in \mathbb{Z}$ . Since  $f$  can have at most countably many points of discontinuity and intervals of constancy, we have  $\Phi_{n,m} = \{\Phi_{n,m}^-, \Phi_{n,m}^+\} \in \mathcal{D}$  almost surely. We call the function  $f$  the *disturbance* and we call  $(\Phi_{n,m} : m, n \in \mathbb{Z}, m \leq n)$  the *discrete disturbance flow*.

Define  $\rho = \rho(f) \in (0, \infty)$  by

$$\rho \int_0^1 \tilde{f}(x)^2 dx = 1. \quad (2)$$

We shall consider scaling limits in time and space. For the time rescaling, it is convenient to embed the discrete-time flow in continuous-time, and in such a way as to normalize the mean square displacement per unit time. We do this in two ways, one related in a more transparent way to the discrete flow, the other sitting more cleanly in continuous time. There is little difference in the arguments needed in their analysis and we shall run them in parallel. Let  $N$  be a Poisson random measure on  $\mathbb{R}$  of intensity  $\rho$  and set

$$N_t = \begin{cases} N(0, t], & t \geq 0, \\ -N(t, 0], & t < 0. \end{cases}$$

Then, for  $s, t \in \mathbb{R}$  with  $s \leq t$ , define

$$\Phi_{(s,t]} = \Phi_{[\rho t], [\rho s]}, \quad \tilde{\Phi}_{(s,t]} = \Phi_{N_t, N_s}.$$

We extend these definitions to intervals closed on the left (respectively, open on the right) by replacing the right-continuous functions  $s \mapsto \lfloor \rho s \rfloor$  and  $s \mapsto N_s$  (respectively,  $t \mapsto \lfloor \rho t \rfloor$  and  $t \mapsto N_t$ ) by their left-continuous modifications. Write  $(\Phi_I : I \subseteq \mathbb{R})$  for the family of maps  $\Phi_I$  where  $I$  ranges over all bounded intervals in  $\mathbb{R}$ . We call  $\Phi = (\Phi_I : I \subseteq \mathbb{R})$  and  $\tilde{\Phi} = (\tilde{\Phi}_I : I \subseteq \mathbb{R})$  respectively the *disturbance flow* and the *Poisson disturbance flow*. In  $\Phi$  the disturbances are applied at times in the lattice  $\mathbb{Z}/\rho$ , whereas the disturbance times in  $\tilde{\Phi}$  are the atoms of  $N$ .

Write  $I = I_1 \oplus I_2$  if  $I_1, I_2$  and  $I$  are intervals with  $I_1 \cap I_2 = \emptyset$  and  $I_1 \cup I_2 = I$ . Note that  $\Phi$  has the following properties and so does  $\tilde{\Phi}$ :

$$\Phi_I^+(x) \text{ and } \Phi_I^-(x) \text{ are random variables for all bounded intervals } I \text{ and all } x \in \mathbb{R}, \quad (3)$$

$$\Phi_I^+ = \Phi_{I_2}^+ \circ \Phi_{I_1}^+ \text{ and } \Phi_I^- = \Phi_{I_2}^- \circ \Phi_{I_1}^- \text{ whenever } I = I_1 \oplus I_2, \quad (4)$$

$$\Phi_{(s,t)}^+(x) = \Phi_{(s,t)}^-(x) = x \text{ for all } x \in \mathbb{R}, \text{ eventually as } s \uparrow t \text{ or } t \downarrow s. \quad (5)$$

We consider a family of diffusive scalings of time and space, which do not preserve the degree 1 property (1). Write  $\bar{\mathcal{D}}$  for the set of all pairs  $\{f^-, f^+\}$  where  $f^+ : \mathbb{R} \rightarrow \mathbb{R}$  is non-decreasing and right-continuous and where  $f^-$  is the left-continuous modification of  $f^+$ . For  $\varepsilon \in (0, 1]$ , define the scaling map  $\sigma_\varepsilon : \mathcal{D} \rightarrow \bar{\mathcal{D}}$  by

$$\sigma_\varepsilon f(x) = \varepsilon^{-1} f(\varepsilon x).$$

Then define

$$\Phi_I^\varepsilon = \sigma_\varepsilon(\Phi_{\varepsilon^2 I}), \quad \tilde{\Phi}_I^\varepsilon = \sigma_\varepsilon(\tilde{\Phi}_{\varepsilon^2 I}).$$

Note that the  $\varepsilon$ -scale disturbance flows  $(\Phi_I^\varepsilon : I \subseteq \mathbb{R})$  and  $(\tilde{\Phi}_I^\varepsilon : I \subseteq \mathbb{R})$  also satisfy (3), (4), (5).

Fix  $e = (s, x) \in \mathbb{R}^2$ , and define two processes  $X_t^{e,-}$  and  $X_t^{e,+}$ , both starting from  $e$ , by setting  $X_t^{e,\pm} = \Phi_{(s,t]}^\pm(x)$  for  $t \geq s$ . Then, almost surely,

$$X_t^{e,-} = X_t^{e,+}, \quad t \geq s. \quad (6)$$

We shall therefore drop the  $\pm$  and write simply  $X^e$ . Define similarly  $\tilde{X}^e$ ,  $X^{e,\varepsilon}$  and  $\tilde{X}^{e,\varepsilon}$ , where we replace  $\Phi$  by  $\tilde{\Phi}$ ,  $\Phi^\varepsilon$  and  $\tilde{\Phi}^\varepsilon$  respectively. Write  $\mu_e^f$ ,  $\tilde{\mu}_e^f$ ,  $\mu_e^{f,\varepsilon}$  and  $\tilde{\mu}_e^{f,\varepsilon}$  for the distributions of these processes on the Skorokhod space  $D_e = D_x([s, \infty), \mathbb{R})$  of cadlag paths starting from  $x$  at time  $s$ . Write  $d_e$  for the Skorokhod metric on  $D_e$  and write  $\mu_e$  for the distribution on  $D_e$  of a standard Brownian motion starting from  $e$ .

**Proposition 2.1.** *We have  $\mu_e^f \rightarrow \mu_e$  and  $\tilde{\mu}_e^f \rightarrow \mu_e$  weakly on  $D_e$ , uniformly in  $f \in \mathcal{D}^*$  as  $\rho(f) \rightarrow \infty$ . Indeed, the following stronger statement holds. We have  $\mu_e^{f,\varepsilon} \rightarrow \mu_e$  and  $\tilde{\mu}_e^{f,\varepsilon} \rightarrow \mu_e$  weakly on  $D_e$ , uniformly in  $f \in \mathcal{D}^*$  and  $\varepsilon \in (0, 1]$  as  $\varepsilon^3 \rho(f) \rightarrow \infty$ .*

*Proof.* We fix  $\varepsilon \in (0, 1]$  and write  $X$  for  $X^{e,\varepsilon}$  and  $\tilde{X}$  for  $\tilde{X}^{e,\varepsilon}$  within the proof to lighten the notation. Note that  $\tilde{X}^\varepsilon$  is a compound Poisson process, making jumps distributed as  $\varepsilon^{-1} \tilde{f}(\Theta_1)$  at rate  $\varepsilon^2 \rho$ . So, for  $t \geq s$ ,

$$\mathbb{E}(\tilde{X}_t - \tilde{X}_s) = \varepsilon \rho(t - s) \int_0^1 \tilde{f}(\theta) d\theta = 0, \quad \mathbb{E}((\tilde{X}_t - \tilde{X}_s)^2) = \rho(t - s) \int_0^1 \tilde{f}(\theta)^2 d\theta = t - s.$$

Hence the processes  $(\tilde{X}_t)_{t \geq s}$  and  $(\tilde{X}_t^2 - t)_{t \geq s}$  are martingales. A standard criterion (see for example [2, page 143] or [11, page 355]) allows us to deduce that the family of laws  $\{\tilde{\mu}_e^{f,\varepsilon} : f \in \mathcal{D}^*, \varepsilon \in (0, 1]\}$  is tight in  $D_e$ . Now  $f$  is non-decreasing so

$$\tilde{f}(\theta) \geq \tilde{f}(\theta_0) - (\theta - \theta_0), \quad \theta \geq \theta_0$$

and so, if  $\tilde{f}(\theta_0) \geq 0$  for some  $\theta_0$ , then

$$\rho^{-1} = \int_0^1 \tilde{f}(\theta)^2 d\theta \geq \int_{\theta_0}^{\theta_0 + \tilde{f}(\theta_0)} (\tilde{f}(\theta_0) - (\theta - \theta_0))^2 d\theta = |\tilde{f}(\theta_0)|^3 / 3$$

and a similar argument leads to the same estimate also when  $\tilde{f}(\theta_0) \leq 0$ . Hence

$$|\tilde{f}(\theta)| \leq (3/\rho)^{1/3}, \quad \theta \in (0, 1]. \quad (7)$$

So the jumps of  $(\tilde{X}_t)_{t \geq s}$  are bounded in absolute value by  $(3/\varepsilon^3 \rho)^{1/3}$ . Let  $\mu$  be any weak limit law for the limit  $\varepsilon^3 \rho(f) \rightarrow \infty$ . Write  $(Z_t)_{t \geq s}$  for the coordinate process on  $D_e$ . Then,

by standard arguments,  $\mu$  is supported on continuous paths and under  $\mu$  both  $(Z_t)_{t \geq s}$  and  $(Z_t^2 - t)_{t \geq s}$  are local martingales in the natural filtration of  $(Z_t)_{t \geq s}$ . Hence  $\mu = \mu_e$  by Lévy's characterization of Brownian motion. The same argument deals with  $\mu_e^{f,\varepsilon}$ , using the fact that  $(X_t)_{t \geq s}$  and  $(X_t^2 - \lfloor \rho \varepsilon^2 t \rfloor / (\rho \varepsilon^2))_{t \geq s}$  are martingales.  $\square$

Given a sequence  $E = (e_k : k \in \mathbb{N})$  in  $\mathbb{R}^2$ , set

$$D_E = \prod_{k=1}^{\infty} D_{e_k}$$

and define a metric  $d_E$  on  $D_E$  by

$$d_E(z, z') = \sum_{k=1}^{\infty} 2^{-k} (d_{e_k}(z_k, z'_k) \wedge 1), \quad z = (z_k : k \in \mathbb{N}), \quad z' = (z'_k : k \in \mathbb{N}).$$

Then  $(D_E, d_E)$  is a complete separable metric space and  $(X^{e_k} : k \in \mathbb{N})$  and  $(\tilde{X}^{e_k} : k \in \mathbb{N})$  are random variables in  $D_E$ . Write  $\mu_E^f$  and  $\tilde{\mu}_E^f$  for their respective distributions on  $D_E$ . Similarly, for  $\varepsilon \in (0, 1]$ , write  $\mu_{E_\varepsilon}^{f,\varepsilon}$  and  $\tilde{\mu}_{E_\varepsilon}^{f,\varepsilon}$  for the distributions on  $D_E$  of the diffusively rescaled families  $(X^{e_{k,\varepsilon}} : k \in \mathbb{N})$  and  $(\tilde{X}^{e_{k,\varepsilon}} : k \in \mathbb{N})$ .

Write  $e_k = (s_k, x_k)$  and denote by  $(Z_t^k)_{t \geq s_k}$  the  $k$ th coordinate process on  $D_E$ , given by  $Z_t^k(z) = z_t^k$ . Consider the filtration  $(\mathcal{Z}_t)_{t \in \mathbb{R}}$  on  $D_E$ , where  $\mathcal{Z}_t$  is the  $\sigma$ -algebra generated by  $(Z_s^k : s_k < s \leq t \vee s_k, k \in \mathbb{N})$ . Write  $C_E$  for the (measurable) subset of  $D_E$  where each coordinate path is continuous. Define on  $C_E$

$$\bar{T}^{jk} = \inf\{t \geq s_j \vee s_k : Z_t^j = Z_t^k\}, \quad T^{jk} = \inf\{t \geq s_j \vee s_k : Z_t^j - Z_t^k \in \mathbb{Z}\}.$$

We sometimes think of the paths  $(Z_t^k)_{t \geq s_k}$  as liftings of paths in the circle  $\mathbb{R}/\mathbb{Z}$ . Then the times  $T^{jk}$  are collision times of the circle-valued paths. The following is a convenient reformulation of a result of Arratia [1], along with a circle-valued variant.

**Proposition 2.2.** *There exists a unique Borel probability measure  $\bar{\mu}_E$  on  $D_E$  under which, for all  $j, k$ , the processes  $(Z_t^k)_{t \geq s_k}$  and  $(Z_t^j Z_t^k - (t - \bar{T}^{jk})^+)_{t \geq s_j \vee s_k}$  are both continuous local martingales in the filtration  $(\mathcal{Z}_t)_{t \in \mathbb{R}}$ .*

*Moreover, there exists a unique Borel probability measure  $\mu_E$  on  $D_E$  under which, for all  $j, k$ , the processes  $(Z_t^k)_{t \geq s_k}$  and  $(Z_t^j Z_t^k - (t - T^{jk})^+)_{t \geq s_j \vee s_k}$  are both continuous local martingales in the filtration  $(\mathcal{Z}_t)_{t \in \mathbb{R}}$ .*

Of course  $\mu_E$  and  $\bar{\mu}_E$  are supported on  $C_E$  and may naturally be considered as measures defined there. We sketch a proof. For existence, one can take independent Brownian motions from each of the given time-space starting points and then impose a rule of coalescence on collision, deleting the path of lower index. The law of the resulting process has the desired

properties. On the other hand, given a probability measure such as described in the proposition, on some larger probability space, one can use a supply of independent Brownian motions to resurrect the paths deleted at each collision. Then Lévy's characterization can be used to see that one has recovered the set-up used for existence. This gives uniqueness.

Consider now a limit in which the basic map  $f$  is an increasingly well localized perturbation of the identity, where we quantify this property in terms of the smallest constant  $\lambda = \lambda(f, \varepsilon) \in (0, 1]$  such that

$$\rho \int_0^1 |\tilde{f}(x+a)\tilde{f}(x)|dx \leq \lambda, \quad a \in [\varepsilon\lambda, 1 - \varepsilon\lambda].$$

Write  $\lambda(f) = \lambda(f, 1)$ .

**Proposition 2.3.** *We have  $\mu_E^f \rightarrow \mu_E$  and  $\tilde{\mu}_E^f \rightarrow \mu_E$  weakly on  $D_E$ , uniformly in  $f \in \mathcal{D}^*$ , as  $\rho(f) \rightarrow \infty$  and  $\lambda(f) \rightarrow 0$ .*

*Moreover  $\mu_E^{f,\varepsilon} \rightarrow \bar{\mu}_E$  and  $\tilde{\mu}_E^{f,\varepsilon} \rightarrow \bar{\mu}_E$  weakly on  $D_E$ , uniformly in  $f \in \mathcal{D}^*$ , as  $\varepsilon \rightarrow 0$  with  $\varepsilon^3 \rho(f) \rightarrow \infty$  and  $\lambda(f, \varepsilon) \rightarrow 0$ .*

*Proof.* We write  $X^k$  for  $X^{e_k}$  and other similar notations within the proof. For each  $k$ , the family of marginal laws  $\{\tilde{\mu}_{e_k}^{f,\varepsilon} : f \in \mathcal{D}^*, \varepsilon \in (0, 1]\}$  is tight, as in Proposition 2.1. Hence the family of laws  $\{\tilde{\mu}_E^{f,\varepsilon} : f \in \mathcal{D}^*, \varepsilon \in (0, 1]\}$  is also tight. Consider first the unscaled Poisson case. Let  $\mu$  be any weak limit law for  $\{\tilde{\mu}_E^f : f \in \mathcal{D}^*\}$  under the limits  $\rho = \rho(f) \rightarrow \infty$  and  $\lambda = \lambda(f) \rightarrow 0$ . For all  $j, k$  the process

$$\tilde{X}_t^j \tilde{X}_t^k - \int_{s_j \vee s_k}^t b(\tilde{X}_s^j, \tilde{X}_s^k) ds, \quad t \geq s_j \vee s_k,$$

is a martingale, where

$$b(x, x') = \rho \int_0^1 \tilde{f}(x-z)\tilde{f}(x'-z)dz.$$

We have  $|b(x, x')| \leq \lambda$  whenever  $\lambda \leq |x - x'| \leq 1 - \lambda$ . Hence, by standard arguments, under  $\mu$ , the process  $(Z_t^j Z_t^k : s_j \vee s_k \leq t < T^{jk})$  is a local martingale. We know from the proof of Proposition 2.1 that, under  $\mu$ , the processes  $(Z_t^j : t \geq s_j)$ ,  $((Z_t^j)^2 - t : t \geq s_j)$  and  $(Z_t^k : t \geq s_k)$  are continuous local martingales. But  $\mu$  inherits from the laws  $\tilde{\mu}_E^f$  the property that, almost surely, for all  $n \in \mathbb{Z}$ , the process  $(Z_t^j - Z_t^k + n : t \geq s_j \vee s_k)$  does not change sign. Hence, by an optional stopping argument,  $Z_t^j - Z_t^k$  is constant for  $t \geq T^{jk}$ . It follows that  $(Z_t^j Z_t^k - (t - T^{jk})^+)_t$  is a continuous local martingale. Hence  $\mu = \mu_E$ , by Proposition 2.2.

Consider now any weak limit  $\nu$  of the family  $\{\tilde{\mu}_E^{f,\varepsilon} : f \in \mathcal{D}^*, \varepsilon \in (0, 1]\}$ , subject to  $\varepsilon \rightarrow 0$  with  $\varepsilon^3 \rho(f) \rightarrow \infty$  and  $\lambda = \lambda(f, \varepsilon) \rightarrow 0$ . By Proposition 2.1, under  $\nu$ , for all  $j$ , the processes



$(Z_t^j : t \geq s_j)$  and  $((Z_t^j)^2 - t : t \geq s_j)$  are continuous local martingales. For all  $j, k$ , the process

$$\tilde{X}_t^{j,\varepsilon} \tilde{X}_t^{k,\varepsilon} - \int_{s_j \vee s_k}^t b(\varepsilon \tilde{X}_s^{j,\varepsilon}, \varepsilon \tilde{X}_s^{k,\varepsilon}) ds, \quad t \geq s_j \vee s_k,$$

is a martingale. Note that  $|b(\varepsilon \tilde{X}_s^{j,\varepsilon}, \varepsilon \tilde{X}_s^{k,\varepsilon})| \leq \lambda$  until  $|\tilde{X}_t^{j,\varepsilon} - \tilde{X}_t^{k,\varepsilon}|$  leaves  $[\lambda, \varepsilon^{-1} - \lambda]$ . Define for  $R \geq 1$

$$T^{jk,R} = \inf\{t \geq s_j \vee s_k : |Z_t^j - Z_t^k| \notin [1/R, R]\}$$

then,  $T^{jk,R} \uparrow \bar{T}^{jk}$  everywhere on  $C_E$ . Under  $\nu$ , the process  $(Z_t^j Z_t^k : s_j \vee s_k \leq t < T^{jk,R})$  is a local martingale for all  $R$ , so  $(Z_t^j Z_t^k : s_j \vee s_k \leq t < \bar{T}^{jk})$  is also a local martingale. Now  $\nu$  inherits from the laws  $\tilde{\mu}_E^{f,\varepsilon}$  the property that, almost surely, the process  $(Z_t^j - Z_t^k : t \geq s_j \vee s_k)$  does not change sign. Hence,  $Z_t^j - Z_t^k$  is constant for  $t \geq \bar{T}^{jk}$ . It follows that  $(Z_t^j Z_t^k - (t - \bar{T}^{jk})^+)_t$  is a continuous local martingale. Hence  $\nu = \bar{\mu}_E$ , by Proposition 2.2.

The same arguments apply in the lattice case, using the martingale

$$X_t^{j,\varepsilon} X_t^{k,\varepsilon} - \frac{1}{\rho \varepsilon^2} \sum_{n=\lfloor \rho \varepsilon^2 (s_j \vee s_k) \rfloor}^{\lfloor \rho \varepsilon^2 t \rfloor - 1} b(\varepsilon X_{n/(\rho \varepsilon^2)}^{j,\varepsilon}, \varepsilon X_{n/(\rho \varepsilon^2)}^{k,\varepsilon}), \quad t \geq s_j \vee s_k.$$

□

### 3 A new state-space for the coalescing Brownian flow

Proposition 2.3 is unsatisfactory in that it expresses convergence of the disturbance flow only for a given sequence of starting points. To remedy this, we must first formulate a suitable limit object. This object is known as Arratia's flow, or the Brownian web, and has been studied in some depth. However, we have found it convenient to introduce a new state-space of flows, which we now describe. We focus for now on the periodic case and discuss the non-periodic case briefly at the end of the section, referring to [4] for a full account of the extensions needed in that case.

We begin by defining a metric on  $\mathcal{D}$ . Let  $\mathcal{S}$  denote the set of all periodic contractions on  $\mathbb{R}$  having period 1. Each  $f \in \mathcal{D}$  can be identified with some  $f^\times \in \mathcal{S}$  by drawing new axes at an angle  $\pi/4$  with the old, and scaling appropriately. See Figure 1. More formally, since  $x + f^+(x)$  is strictly increasing in  $x$ , there is for each  $t \in \mathbb{R}$  a unique  $x \in \mathbb{R}$  such that

$$\frac{x + f^-(x)}{2} \leq t \leq \frac{x + f^+(x)}{2}.$$

Define  $f^\times(t) = t - x$ . Note that  $\text{id}^\times = 0$ . Then the map  $f \mapsto f^\times : \mathcal{D} \rightarrow \mathcal{S}$  is a bijection, so we can define a metric  $d_{\mathcal{D}}$  on  $\mathcal{D}$  by

$$d_{\mathcal{D}}(f, g) = \|f^\times - g^\times\| = \sup_{t \in [0,1]} |f^\times(t) - g^\times(t)|.$$



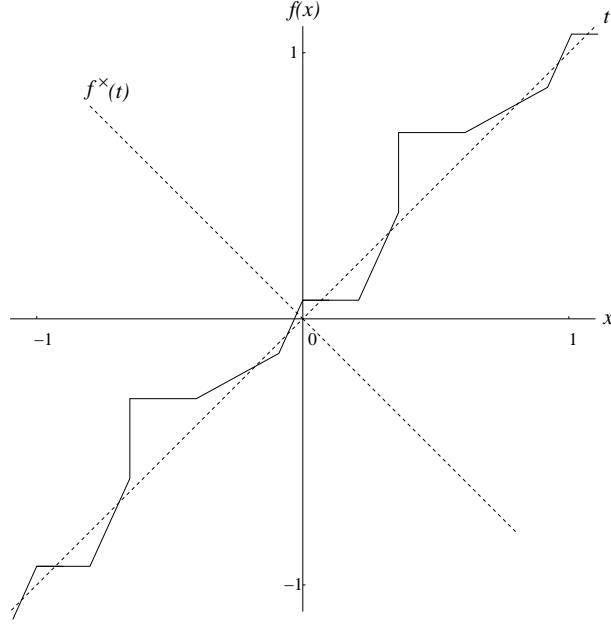


Figure 1: The map  $f^\times$  obtained from  $f$  by rotating the axes by  $\frac{\pi}{4}$ .

A proof of the italicized assertion is given in the Appendix. The same is true for some further technical assertions which will be made below, written also in italics. The metric space  $(\mathcal{S}, \|\dots\|)$  is complete and locally compact, so the same is true for  $(\mathcal{D}, d_{\mathcal{D}})$ . An alternative characterization<sup>5</sup> of the metric  $d_{\mathcal{D}}$  is as follows: *for  $f, g \in \mathcal{D}$  and  $\varepsilon > 0$ , we have*

$$d_{\mathcal{D}}(f, g) \leq \varepsilon \iff f^-(x - \varepsilon) - \varepsilon \leq g^-(x) \leq g^+(x) \leq f^+(x + \varepsilon) + \varepsilon \text{ for all } x \in \mathbb{R}.$$

We deduce that, for  $f, g \in \mathcal{D}$ ,

$$d_{\mathcal{D}}(f, g) \leq \|f - g\|, \quad 2d_{\mathcal{D}}(f, \text{id}) = \|f - \text{id}\|,$$

and

$$d_{\mathcal{D}}(f, g \circ f) \leq \|g - \text{id}\| \text{ when } g \circ f \in \mathcal{D}, \quad d_{\mathcal{D}}(f, f \circ g) \leq \|g - \text{id}\| \text{ when } f \circ g \in \mathcal{D}.$$

Moreover, *for any sequence  $(f_n : n \in \mathbb{N})$  in  $\mathcal{D}$ ,*

$$f_n \rightarrow f \iff f_n(x) \rightarrow f(x) \text{ at every point } x \text{ where } f \text{ is continuous.}$$

---

<sup>5</sup>Thus,  $d_{\mathcal{D}}$  is a close relative of the Lévy metric sometimes used on the set of distribution functions for real random variables. The relationships of such a metric to the operations of composition and inversion in  $\mathcal{D}$ , which are significant for us, do not appear to have been studied.

Here and below, we write  $f_n \rightarrow f$  to mean convergence in the metric  $d_{\mathcal{D}}$ .

We now define our space of flows. We call them weak flows to emphasise that the usual flow property may fail at points of spatial discontinuity. Consider  $\phi = (\phi_{ts} : s, t \in \mathbb{R}, s < t)$ , with  $\phi_{ts} \in \mathcal{D}$  for all  $s, t$ . Say that  $\phi$  is a *weak flow* if

$$\phi_{ut}^- \circ \phi_{ts}^- \leq \phi_{us}^- \leq \phi_{us}^+ \leq \phi_{ut}^+ \circ \phi_{ts}^+, \quad s < t < u.$$

Say that  $\phi$  is *continuous* if, for all  $t \in \mathbb{R}$ ,

$$\phi_{ts} \rightarrow \text{id} \text{ as } s \uparrow t, \quad \phi_{ut} \rightarrow \text{id} \text{ as } u \downarrow t.$$

Write  $C^\circ(\mathbb{R}, \mathcal{D})$  for the set of all continuous weak flows. It will be convenient sometimes to extend a continuous weak flow  $\phi$  to the diagonal, which we do by setting  $\phi_{ss} = \text{id}$  for all  $s \in \mathbb{R}$ . Then, for any  $\phi \in C^\circ(\mathbb{R}, \mathcal{D})$ , the map

$$(s, t) \mapsto \phi_{ts} : \{(s, t) : s \leq t\} \rightarrow \mathcal{D} \quad (8)$$

is continuous.

Define, for  $\phi, \psi \in C^\circ(\mathbb{R}, \mathcal{D})$ ,

$$d_C(\phi, \psi) = \sum_{n=1}^{\infty} 2^{-n} \{d_C^{(n)}(\phi, \psi) \wedge 1\},$$

where

$$d_C^{(n)}(\phi, \psi) = \sup_{s, t \in (-n, n), s < t} d_{\mathcal{D}}(\phi_{ts}, \psi_{ts}).$$

Then  $d_C$  is a metric on  $C^\circ(\mathbb{R}, \mathcal{D})$ , under which  $C^\circ(\mathbb{R}, \mathcal{D})$  is complete and separable. Define, for  $e = (s, x) \in \mathbb{R}^2$  and  $t \geq s$ , evaluation maps  $Z_t^{e,+}$  and  $Z_t^{e,-}$  on  $C^\circ(\mathbb{R}, \mathcal{D})$  by

$$Z_t^{e,\pm}(\phi) = \phi_{ts}^\pm(x).$$

Then, for all  $\phi \in C^\circ(\mathbb{R}, \mathcal{D})$ , the maps  $t \mapsto Z_t^{e,\pm}(\phi) : [s, \infty) \rightarrow \mathbb{R}$  are continuous. So we can consider the processes  $Z^{e,\pm} = (Z_t^{e,\pm} : t \geq s)$  as  $C_e$ -valued random variables on  $C^\circ(\mathbb{R}, \mathcal{D})$ . Write  $Z^e = Z^{e,+}$  to lighten the notation. Define a  $\sigma$ -algebra  $\mathcal{F}$  and a filtration  $(\mathcal{F}_t)_{t \in \mathbb{R}}$  on  $C^\circ(\mathbb{R}, \mathcal{D})$  by

$$\mathcal{F} = \sigma(Z_t^e : e \in \mathbb{R}^2, t \geq s(e)), \quad \mathcal{F}_t = \sigma(Z_r^e : e \in \mathbb{R}^2, r \in (-\infty, t] \cap [s(e), \infty)).$$

Then  $\mathcal{F}_t$  is generated by the random variables  $Z_r^e$  with  $e \in \mathbb{Q}^2$  and  $r \in (-\infty, t] \cap [s(e), \infty)$ , and  $\mathcal{F}$  is the Borel  $\sigma$ -algebra of the metric  $d_C$ . Define for  $e = (s, x)$  and  $e' = (s', x')$  the collision time  $T^{ee'} : C^\circ(\mathbb{R}, \mathcal{D}) \rightarrow [0, \infty]$  by

$$T^{ee'}(\phi) = \inf\{t \geq s \vee s' : Z_t^e(\phi) - Z_t^{e'}(\phi) \in \mathbb{Z}\}.$$

The following result translates into the language of continuous weak flows a result of Tóth and Werner [18, Theorem 2.1], which itself was a variant of a result of Arratia [1]. We shall give a complete proof, in part because we need most components of the proof also for our main convergence result, and in part because our framework leads to some simplifications, for example in the probabilistic underpinnings contained in Proposition 8.10. The formulation in terms of continuous weak flows has advantages in leading to a unique object, with a natural time-reversal invariance, and for the derivation of weak limits.

**Theorem 3.1.** *There exists a unique Borel probability measure  $\mu_A$  on  $C^\circ(\mathbb{R}, \mathcal{D})$  under which, for all  $e, e' \in \mathbb{R}^2$ , the processes  $(Z_t^e)_{t \geq s(e)}$  and  $(Z_t^e Z_t^{e'} - (t - T^{ee'})^+)_{t \geq s(e) \vee s(e')}$  are continuous local martingales for  $(\mathcal{F}_t)_{t \in \mathbb{R}}$ . Moreover, for all  $e \in \mathbb{R}^2$  we have  $\mu_A$ -almost surely  $Z^{e,+} = Z^{e,-}$ .*

*Proof.* Fix an enumeration  $E = (e_k : k \in \mathbb{N})$  of  $\mathbb{Q}^2$ . Define  $Z^{E,\pm} : C^\circ(\mathbb{R}, \mathcal{D}) \rightarrow C_E$  by  $Z^{E,\pm}(\phi) = (Z^{e_k,\pm}(\phi) : k \in \mathbb{N})$ . Then, we have  $\mathcal{F}_t = \{(Z^{E,+})^{-1}(B) : B \in \mathcal{Z}_t\}$ . Set

$$C_E^{\circ,\pm} = \{Z^{E,\pm}(\phi) : \phi \in C^\circ(\mathbb{R}, \mathcal{D})\}.$$

Then the sets  $C_E^{\circ,\pm}$  are measurable subsets of  $C_E$  with  $\mu_E(C_E^{\circ,\pm}) = 1$ . Moreover  $Z^{E,\pm}$  maps  $C^\circ(\mathbb{R}, \mathcal{D})$  bijectively to  $C_E^{\circ,\pm}$  and the inverse bijections  $C_E^{\circ,\pm} \rightarrow C^\circ(\mathbb{R}, \mathcal{D})$ , which we denote by  $\Phi^{E,\pm}$ , are measurable. Write  $Z^E$  for  $Z^{E,+}$  and  $\Phi^E$  for  $\Phi^{E,+}$ . Then, on  $C_E^{\circ,+}$ , for all  $j, k \in \mathbb{N}$ , we have

$$Z^{e_k} \circ \Phi^E = Z^k, \quad T^{e_j e_k} \circ \Phi^E = T^{jk}$$

and for all  $t \in \mathbb{R}$  and  $B \in \mathcal{F}_t$  we have  $1_B \circ \Phi^E = 1_{B'}$  for some  $B' \in \mathcal{Z}_t$ . Define  $\mu_A = \mu_E \circ (\Phi^E)^{-1}$ . By Proposition 2.2, under  $\mu_A$ , for all  $j, k \in \mathbb{N}$ , the processes  $(Z_t^{e_k})_{t \geq s_k}$  and  $(Z_t^{e_j} Z_t^{e_k} - (t - T^{e_j e_k})^+)_{t \geq s_j \vee s_k}$  are continuous local martingales for  $(\mathcal{F}_t)_{t \in \mathbb{R}}$ .

On the other hand, for any probability measure  $\mu$  on  $C^\circ(\mathbb{R}, \mathcal{D})$  having this property, under the image measure  $\mu \circ (Z^E)^{-1}$  on  $C_E$ , for all  $j, k \in \mathbb{N}$ , the processes  $(Z_t^k)_{t \geq s_k}$  and  $(Z_t^j Z_t^k - (t - T^{jk})^+)_{t \geq s_j \vee s_k}$  are continuous local martingales for  $(\mathcal{Z}_t)_{t \in \mathbb{R}}$ , so  $\mu \circ (Z^E)^{-1} = \mu_E$  by Proposition 2.2, and so  $\mu = \mu_A$ .

Given  $e, e' \in \mathbb{R}^2$ , all the assertions above hold also when  $E$  is replaced by the sequence  $E' = (e, e', e_1, e_2, \dots)$ . We repeat the steps taken to obtain a probability measure  $\mu'_A = \mu_{E'} \circ (\Phi^{E'})^{-1}$  on  $C^\circ(\mathbb{R}, \mathcal{D})$ . Then, under  $\mu'_A$ , the processes  $(Z_t^e)_{t \geq s(e)}$  and  $(Z_t^e Z_t^{e'} - (t - T^{ee'})^+)_{t \geq s(e) \vee s(e')}$  are continuous local martingales for  $(\mathcal{F}_t)_{t \in \mathbb{R}}$ . But also, under  $\mu'_A$ , for all  $j, k \in \mathbb{N}$ , the processes  $(Z_t^{e_k})_{t \geq s_k}$  and  $(Z_t^{e_j} Z_t^{e_k} - (t - T^{e_j e_k})^+)_{t \geq s_j \vee s_k}$  are continuous local martingales for  $(\mathcal{F}_t)_{t \in \mathbb{R}}$ , so  $\mu_A = \mu'_A$ .

Finally, we have  $\Phi^{E',+} = \Phi^{E',-}$  on  $C_{E'}^{\circ,-} \cap C_{E'}^{\circ,+}$ , so  $Z^{e,-}(\Phi^{E'}) = Z^{e,-}(\Phi^{E',-}) = Z^{e,+}(\Phi^{E'})$ ,  $\mu_{E'}$ -almost surely, and so  $Z^{e,-} = Z^{e,+}$ ,  $\mu_A$ -almost surely, as claimed.  $\square$

We call any  $C^\circ(\mathbb{R}, \mathcal{D})$ -valued random variable with law  $\mu_A$  a *coalescing Brownian flow on the circle*. The notions needed for the non-periodic case are closely analogous, provided we

generalize  $d_{\mathcal{D}}$  by a metric on  $\bar{\mathcal{D}}$  obtained, via the bijection  $f \mapsto f^\times$ , from a metric of uniform convergence on compacts for contractions on  $\mathbb{R}$ . We denote by  $C^\circ(\mathbb{R}, \bar{\mathcal{D}})$  the set of continuous weak flows with values in  $\bar{\mathcal{D}}$ . The coordinate processes  $Z^e = Z^{e,+}$  and  $Z^{e,-}$  and the filtration  $(\mathcal{F}_t)_{t \in \mathbb{R}}$  are defined for  $C^\circ(\mathbb{R}, \bar{\mathcal{D}})$  just as for  $C^\circ(\mathbb{R}, \mathcal{D})$ , but we define now for  $e = (s, x)$  and  $e' = (s', x')$  the collision time  $\bar{T}^{ee'} : C^\circ(\mathbb{R}, \bar{\mathcal{D}}) \rightarrow [0, \infty]$  by

$$\bar{T}^{ee'}(\phi) = \inf\{t \geq s \vee s' : Z_t^e(\phi) = Z_t^{e'}(\phi)\}.$$

The following result is proved in [4, Section 9].

**Theorem 3.2.** *There exists a unique Borel probability measure  $\bar{\mu}_A$  on  $C^\circ(\mathbb{R}, \bar{\mathcal{D}})$  under which, for all  $e, e' \in \mathbb{R}^2$ , the processes  $(Z_t^e)_{t \geq s(e)}$  and  $(Z_t^e Z_t^{e'} - (t - \bar{T}^{ee'})^+)_{t \geq s(e) \vee s(e')}$  are continuous local martingales for  $(\mathcal{F}_t)_{t \in \mathbb{R}}$ . Moreover, for all  $e \in \mathbb{R}^2$  we have  $\bar{\mu}_A$ -almost surely  $Z^{e,+} = Z^{e,-}$ .*

A  $C^\circ(\mathbb{R}, \bar{\mathcal{D}})$ -valued random variable with law  $\bar{\mu}_A$  is called a *coalescing Brownian flow*. The relationship of such a random variable to the Brownian web [6, 7] is explored in the Appendix.

## 4 Complete coalescence time

In this section we digress to discuss the complete coalescence time  $T$  of a coalescing Brownian flow  $\Phi$  on the circle, given by

$$T = \inf\{t \geq 0 : \Phi_{t0}^+(x) = y + n \text{ for some } n \in \mathbb{Z}, \text{ for all } x \in \mathbb{R}, \text{ for some } y \in \mathbb{R}\}.$$

It is known by an indirect argument, see [3], that

$$\mathbb{E}(e^{\lambda T}) = \sqrt{\lambda} / \sin \sqrt{\lambda}, \quad \lambda < \pi^2. \quad (9)$$

We will give an alternative and direct proof.

Fix  $N \in \mathbb{N}$  and define for  $t \geq 0$

$$B_t^k = \phi_{t0}(k/N) - \phi_{t0}((k-1)/N), \quad k = 1, \dots, N.$$

Then each process  $B^k$  is a Brownian motion of diffusivity 2, starting from  $1/N$  and stopped on hitting 0 or 1. Consider the stopping time  $S = \inf\{t \geq 0 : B_t^k = 1 \text{ for some } k\}$  and note that  $B_S^k = 0$  for all but one random value,  $k = K$  say, for which  $B_S^K = 1$ . Define

$$M_t = M_t^{(N)} = e^{\lambda t} \sum_{k=1}^N \sin \left\{ \sqrt{\lambda} B_t^k \right\}$$

then the stopped process  $M^S$  is a martingale so, for all  $t \geq 0$ ,

$$N \sin \left\{ \sqrt{\lambda}/N \right\} = M_0 = \mathbb{E}(M_{S \wedge t}) = \mathbb{E} \left( e^{\lambda(S \wedge t)} \sum_{k=1}^N \sin \left\{ \sqrt{\lambda} B_{S \wedge t}^k \right\} \right) \geq \mathbb{E}(e^{\lambda(S \wedge n)}) \sin \sqrt{\lambda}.$$

For  $\lambda < \pi^2$  the final inequality allows us to see that  $\mathbb{E}(e^{\lambda S}) < \infty$ , so we can let  $t \rightarrow \infty$  to obtain

$$N \sin \{ \sqrt{\lambda}/N \} = \mathbb{E}(e^{\lambda S}) \sin \sqrt{\lambda}.$$

On letting  $N \rightarrow \infty$  we obtain (9).

In fact, it is not hard to see that  $M_t^{(N)}$  increases with  $N$  for all  $t \geq 0$  and is eventually constant for all  $t > 0$ . The limit process  $\bar{M}$  is also a martingale with  $\bar{M}_0 = \sqrt{\lambda}$  and  $\bar{M}_T = e^{\lambda T} \sin \sqrt{\lambda}$ , and the optional stopping argument can alternatively be applied directly to  $\bar{M}$ .

From (9) we can identify  $T$  as having the same law as one half of the time  $\tilde{T}$  taken for a BES(3) to get from 0 to 1. This can also be seen directly using the relation

$$S = \sum_{k=1}^N S_k 1_{\{B^k(S_k)=1\}}$$

where  $S_k = \inf\{t \geq 0 : B_t^k \in \{0, 1\}\}$ . Then, for any bounded measurable function  $f$ ,

$$\mathbb{E}(f(S)) = \sum_{k=1}^N \mathbb{E}(f(S_k) 1_{\{B^k(S_k)=1\}}) = \mathbb{E}(f(S_1) | B^1(S_1) = 1)$$

and, on letting  $N \rightarrow \infty$ , we obtain  $\mathbb{E}(f(T)) = \mathbb{E}(f(\tilde{T}/2))$ . We thank Neil O'Connell and Marc Yor for this observation.

## 5 A Skorokhod-type space of non-decreasing flows on the circle

Since the disturbance flow is not continuous in time, it will be necessary to introduce a larger flow space to accommodate it. Consider now  $\phi = (\phi_I : I \subseteq \mathbb{R})$ , where  $\phi_I \in \mathcal{D}$  and  $I$  ranges over all non-empty bounded intervals. Recall that we write  $I = I_1 \oplus I_2$  if  $I, I_1, I_2$  are intervals with  $I_1 \cap I_2 = \emptyset$  and  $I_1 \cup I_2 = I$ . Say that  $\phi$  is a *weak flow* if,

$$\phi_{I_2}^- \circ \phi_{I_1}^- \leq \phi_I^- \leq \phi_I^+ \leq \phi_{I_2}^+ \circ \phi_{I_1}^+, \quad I = I_1 \oplus I_2. \quad (10)$$

Say that  $\phi$  is *cadlag*<sup>6</sup> if, for all  $t \in \mathbb{R}$ ,

$$\phi_{(s,t)} \rightarrow \text{id} \quad \text{as } s \uparrow t, \quad \phi_{(t,u)} \rightarrow \text{id} \quad \text{as } u \downarrow t.$$

Write  $D^\circ(\mathbb{R}, \mathcal{D})$  for the set of cadlag weak flows. It will be convenient to extend a cadlag weak flow  $\phi$  to the empty interval by setting  $\phi_\emptyset = \text{id}$ . Given a bounded interval  $I$  and a sequence of bounded intervals  $(I_n : n \in \mathbb{N})$ , write  $I_n \rightarrow I$  if

$$I = \bigcup_n \bigcap_{m \geq n} I_m = \bigcap_n \bigcup_{m \geq n} I_m.$$

For any  $\phi \in D^\circ(\mathbb{R}, \mathcal{D})$ , we have

$$\phi_{I_n} \rightarrow \phi_I \quad \text{as } I_n \rightarrow I. \quad (11)$$

Let  $\phi$  be a cadlag weak flow and suppose that  $\phi_{\{t\}} = \text{id}$  for all  $t \in \mathbb{R}$ . Then, using (10), we have  $\phi_{(s,t)} = \phi_{(s,t]} = \phi_{[s,t)} = \phi_{[s,t]}$  for all  $s < t$  and, denoting all these functions by  $\phi_{ts}$ , the family  $(\phi_{ts} : s, t \in \mathbb{R}, s < t)$  is a continuous weak flow in the sense of the preceding section.

For  $\phi, \psi \in D^\circ(\mathbb{R}, \mathcal{D})$  and  $n \geq 1$ , define

$$d_D^{(n)}(\phi, \psi) = \inf_\lambda \left\{ \gamma(\lambda) \vee \sup_{I \subseteq \mathbb{R}} \|\chi_n(I)\phi_I^\times - \chi_n(\lambda(I))\psi_{\lambda(I)}^\times\| \right\},$$

where the infimum is taken over the set of increasing homeomorphisms  $\lambda$  of  $\mathbb{R}$ , where

$$\gamma(\lambda) = \sup_{t \in \mathbb{R}} |\lambda(t) - t| \vee \sup_{s, t \in \mathbb{R}, s < t} \left| \log \left( \frac{\lambda(t) - \lambda(s)}{t - s} \right) \right|,$$

and where  $\chi_n$  is the cutoff function<sup>7</sup> given by

$$\chi_n(I) = 0 \vee (n + 1 - R) \wedge 1, \quad R = \sup I \vee (-\inf I).$$

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<sup>6</sup>This definition is more symmetric in time than is usual for ‘cadlag’: a more accurate acronym would be *laglad*

<sup>7</sup>As in the case of the standard Skorokhod topology, localization in time sits awkwardly with the stretching of time introduced via the homeomorphisms  $\lambda$ . There is no fundamental obstacle, just some messiness at the edges. Note that, when  $I \cup \lambda(I) \subseteq [-n, n]$ , we have

$$\|\chi_n(I)\phi_I^\times - \chi_n(\lambda(I))\psi_{\lambda(I)}^\times\| = d_{\mathcal{D}}(\phi_I, \psi_{\lambda(I)}).$$

Also, for all intervals  $I$ , we have  $|\chi_n(\lambda(I)) - \chi_n(I)| \leq \gamma(\lambda)$  and

$$\|\chi_n(I)\phi_I^\times - \chi_n(\lambda(I))\psi_{\lambda(I)}^\times\| \leq \chi_n(I)d_{\mathcal{D}}(\phi_I, \psi_{\lambda(I)}) + |\chi_n(\lambda(I)) - \chi_n(I)|\|\psi_{\lambda(I)}^\times\|.$$

Then define

$$d_D(\phi, \psi) = \sum_{n=1}^{\infty} 2^{-n} \{d_D^{(n)}(\phi, \psi) \wedge 1\}. \quad (12)$$

Then  $d_D$  is a metric on  $D^\circ(\mathbb{R}, \mathcal{D})$  under which  $D^\circ(\mathbb{R}, \mathcal{D})$  is complete and separable. Moreover the metrics  $d_C$  and  $d_D$  generate the same topology on  $C^\circ(\mathbb{R}, \mathcal{D})$ . For the metric  $d_D$ , for all bounded intervals  $I$  and all  $x \in \mathbb{R}$ , the evaluation map

$$\phi \mapsto \phi_I^+(x) : D^\circ(\mathbb{R}, \mathcal{D}) \rightarrow \mathbb{R}$$

is Borel measurable. Moreover the Borel  $\sigma$ -algebra on  $D^\circ(\mathbb{R}, \mathcal{D})$  is generated by the set of all such evaluation maps with  $I = (s, t]$  and  $s, t$  and  $x$  rational.

For  $e = (s, x) \in \mathbb{R}^2$  and  $\phi \in D^\circ(\mathbb{R}, \mathcal{D})$ , the maps

$$t \mapsto \phi_{(s,t]}^\pm(x) : [s, \infty) \rightarrow \mathbb{R}$$

are cadlag. Hence we can extend the maps  $Z^e = Z^{e,+}$  and  $Z^{e,-}$ , which we defined on  $C^\circ(\mathbb{R}, \mathcal{D})$  in Section 3, to measurable maps  $Z^{e,\pm} : D^\circ(\mathbb{R}, \mathcal{D}) \rightarrow D_e$  by setting

$$Z^{e,\pm}(\phi) = (\phi_{(s,t]}^\pm(x) : t \geq s).$$

Let  $E = (e_k : k \in \mathbb{N})$  be an enumeration of  $\mathbb{Q}^2$ . Write  $Z^{E,\pm}$  for the maps  $D^\circ(\mathbb{R}, \mathcal{D}) \rightarrow D_E$  given by  $Z^{E,\pm} = (Z^{e_k,\pm} : k \in \mathbb{N})$ . Write  $Z^E = Z^{E,+}$ . The following result is a criterion for weak convergence on  $D^\circ(\mathbb{R}, \mathcal{D})$ .

**Theorem 5.1.** *Let  $(\mu_n : n \in \mathbb{N})$  be a sequence of Borel probability measures on  $D^\circ(\mathbb{R}, \mathcal{D})$  and let  $\mu$  be a Borel probability measure on  $C^\circ(\mathbb{R}, \mathcal{D})$ . Assume that  $Z^{E,-} = Z^{E,+}$  holds  $\mu_n$ -almost surely for all  $n$  and  $\mu$ -almost surely. Assume further that  $\mu_n \circ (Z^E)^{-1} \rightarrow \mu \circ (Z^E)^{-1}$  weakly on  $D_E$ . Then  $\mu_n \rightarrow \mu$  weakly on  $D^\circ(\mathbb{R}, \mathcal{D})$ .*

*Proof.* Set

$$D^\circ(E) = \{\phi \in D^\circ(\mathbb{R}, \mathcal{D}) : Z^{E,+}(\phi) = Z^{E,-}(\phi)\}, \quad D_E^\circ = \{Z^E(\phi) : \phi \in D^\circ(E)\}.$$

Let  $\Phi_n$  and  $\Phi$  be random variables in  $D^\circ(\mathbb{R}, \mathcal{D})$  having distributions  $\mu_n$  and  $\mu$  respectively. Then  $Z^E(\Phi_n) \rightarrow Z^E(\Phi)$  weakly on  $D_E$ . Also  $\Phi_n \in D^\circ(E)$  almost surely and  $\Phi \in D^\circ(E)$  almost surely. So  $Z^E(\Phi_n) \in D_E^\circ$  almost surely and  $Z^E(\Phi) \in C_E^\circ = C_E \cap D_E^\circ$  almost surely. Now  $D_E^\circ$  is measurable and  $Z^E$  maps  $D^\circ(E)$  bijectively to  $D_E^\circ$ . Denote the inverse bijection by  $\Phi^E$ . Then  $\Phi^E : D_E^\circ \rightarrow D^\circ(E)$  is measurable and is continuous at  $C_E^\circ$ . Hence  $\Phi_n = \Phi^E(Z^E(\Phi_n)) \rightarrow \Phi^E(Z^E(\Phi)) = \Phi$  weakly on  $D^\circ(\mathbb{R}, \mathcal{D})$ .  $\square$



In the absence of periodicity, we define analogously the space  $D^\circ(\mathbb{R}, \bar{\mathcal{D}})$  of cadlag weak flows  $(\phi_I : I \subseteq \mathbb{R})$  with  $\phi_I \in \bar{\mathcal{D}}$  for all  $I$ . The Skorokhod-type metric on  $D^\circ(\mathbb{R}, \bar{\mathcal{D}})$  is defined just as for  $D^\circ(\mathbb{R}, \mathcal{D})$ , except that the uniform metric on  $\mathcal{S}$  is replaced by a metric of uniform convergence on compacts on the space  $\bar{\mathcal{S}}$  of contractions on  $\mathbb{R}$ . The following result is proved in [4, Lemma 14.1]

**Theorem 5.2.** *Let  $(\mu_n : n \in \mathbb{N})$  be a sequence of Borel probability measures on  $D^\circ(\mathbb{R}, \bar{\mathcal{D}})$  and let  $\mu$  be a Borel probability measure on  $C^\circ(\mathbb{R}, \bar{\mathcal{D}})$ . Assume that  $Z^{E,-} = Z^{E,+}$  holds  $\mu_n$ -almost surely for all  $n$  and  $\mu$ -almost surely. Assume further that, for any finite sequence  $E$  in  $\mathbb{R}^2$ , we have  $\mu_n \circ (Z^E)^{-1} \rightarrow \mu \circ (Z^E)^{-1}$  weakly on  $D_E$ . Then  $\mu_n \rightarrow \mu$  weakly on  $D^\circ(\mathbb{R}, \bar{\mathcal{D}})$ .*

## 6 Convergence of the disturbance flow to the coalescing Brownian flow

The disturbance flow  $\Phi = (\Phi_I : I \subseteq \mathbb{R})$  and the Poisson disturbance flow  $\tilde{\Phi} = (\tilde{\Phi}_I : I \subseteq \mathbb{R})$  associated to a given basic map  $f \in \mathcal{D}^*$  were defined in Section 2, along with their diffusive rescalings  $\Phi^\varepsilon = (\sigma_\varepsilon(\Phi_{\varepsilon^2 I}) : I \subseteq \mathbb{R})$  and  $\tilde{\Phi}^\varepsilon = (\sigma_\varepsilon(\tilde{\Phi}_{\varepsilon^2 I}) : I \subseteq \mathbb{R})$ . Properties (3), (4) and (5) imply that  $\Phi$  and  $\tilde{\Phi}$  are well-defined Borel random variables in  $D^\circ(\mathbb{R}, \mathcal{D})$ , whilst  $\Phi^\varepsilon$  and  $\tilde{\Phi}^\varepsilon$  are Borel random variables in  $D^\circ(\mathbb{R}, \bar{\mathcal{D}})$  for all  $\varepsilon \in (0, 1]$ . Denote by  $\mu_A^f$  and  $\tilde{\mu}_A^f$  the respective laws of  $\Phi$  and  $\tilde{\Phi}$  on  $D^\circ(\mathbb{R}, \mathcal{D})$ . Denote by  $\mu_A^{f,\varepsilon}$  and  $\tilde{\mu}_A^{f,\varepsilon}$  the laws of  $\Phi^\varepsilon$  and  $\tilde{\Phi}^\varepsilon$  on  $D^\circ(\mathbb{R}, \bar{\mathcal{D}})$ . Note that, by (6), under any of these measures, for all  $e \in \mathbb{R}^2$ , we have  $Z^{e,-} = Z^{e,+}$  almost surely. The same was shown in Theorems 3.1 and 3.2 for the laws  $\mu_A$  and  $\bar{\mu}_A$  of coalescing Brownian flows on  $C^\circ(\mathbb{R}, \mathcal{D})$  and  $C^\circ(\mathbb{R}, \bar{\mathcal{D}})$ . Our main result now follows directly from Proposition 2.3 and Theorems 5.1 and 5.2.

**Theorem 6.1.** *The disturbance flow and the Poisson disturbance flow converge weakly to the coalescing Brownian flow on the circle in the limit as the basic disturbance becomes small and localized. Thus,  $\mu_A^f \rightarrow \mu_A$  and  $\tilde{\mu}_A^f \rightarrow \mu_A$  weakly on  $D^\circ(\mathbb{R}, \mathcal{D})$ , uniformly in  $f \in \mathcal{D}^*$ , as  $\rho(f) \rightarrow \infty$  and  $\lambda(f) \rightarrow 0$ .*

*Moreover, in the same limiting regime, the  $\varepsilon$ -scale disturbance flow and the  $\varepsilon$ -scale Poisson disturbance flow converge weakly to the coalescing Brownian flow (on the line), provided  $\varepsilon \rightarrow 0$  but not too fast relative to the smallness and locality of the basic disturbance. More precisely,  $\mu_A^{f,\varepsilon} \rightarrow \bar{\mu}_A$  and  $\tilde{\mu}_A^{f,\varepsilon} \rightarrow \bar{\mu}_A$  weakly on  $D^\circ(\mathbb{R}, \bar{\mathcal{D}})$ , uniformly in  $f \in \mathcal{D}^*$ , as  $\varepsilon \rightarrow 0$  with  $\varepsilon^3 \rho(f) \rightarrow \infty$  and  $\lambda(f, \varepsilon) \rightarrow 0$ .*

## 7 Time reversal

For  $f^+ \in \mathcal{R}$  and  $f^- \in \mathcal{L}$ , we define a *left-continuous inverse*  $(f^+)^{-1} \in \mathcal{L}$  and a *right-continuous inverse*  $(f^-)^{-1} \in \mathcal{R}$  by

$$(f^+)^{-1}(y) = \inf\{x \in \mathbb{R} : f^+(x) > y\}, \quad (f^-)^{-1}(y) = \sup\{x \in \mathbb{R} : f^-(x) < y\}.$$

The map  $f^+ \mapsto (f^+)^{-1} : \mathcal{R} \rightarrow \mathcal{L}$  is a bijection, with  $((f^+)^{-1})^{-1} = f^+$  and

$$(f_1^+ \circ f_2^+)^{-1} = (f_2^+)^{-1} \circ (f_1^+)^{-1}, \quad f_1, f_2 \in \mathcal{R}.$$

We have  $f^+ \circ (f^+)^{-1} = \text{id}$  if and only if  $f^+$  is a homeomorphism. Define for  $f = \{f^-, f^+\} \in \mathcal{D}$  the *inverse*  $f^{-1} = \{(f^+)^{-1}, (f^-)^{-1}\} \in \mathcal{D}$ . Note that  $(f^{-1})^\times = -f^\times$ , so the map  $f \mapsto f^{-1} : \mathcal{D} \rightarrow \mathcal{D}$  is an isometry.

Define the *time-reversal map*  $\wedge : D^\circ(\mathbb{R}, \mathcal{D}) \rightarrow D^\circ(\mathbb{R}, \mathcal{D})$  by

$$\hat{\phi}_I = \phi_{-I}^{-1}$$

where  $-I = \{-x : x \in I\}$ . It is straightforward to check that this is a well-defined isometry of  $D^\circ(\mathbb{R}, \mathcal{D})$ , which restricts to an isometry of  $C^\circ(\mathbb{R}, \mathcal{D})$ . Define, for  $f \in \mathcal{D}^*$ ,

$$\hat{\mu}_A^f = \mu_A^f \circ \wedge^{-1}.$$

**Proposition 7.1.** *The time-reversal of a disturbance flow with disturbance  $f$  is a disturbance flow with disturbance  $f^{-1}$ . Thus,  $f^{-1} \in \mathcal{D}^*$  and  $\hat{\mu}_A^f = \mu_A^{f^{-1}}$ .*

*Proof.* Fix  $f \in \mathcal{D}^*$ . Set  $g = f^{-1}$  and

$$\Delta = \{(x, y) \in \mathbb{R}^2 : y < f(x)\} = \{(x, y) \in \mathbb{R}^2 : x > g(y)\}, \quad \Delta_0 = \{(x, y) \in \mathbb{R}^2 : y < x\}.$$

Then, by Fubini's theorem,

$$\int_0^1 \tilde{f}(x) dx = \int_0^1 \int_{\mathbb{R}} (1_\Delta - 1_{\Delta_0})(x, y) dx dy = - \int_0^1 \tilde{g}(y) dy$$

and

$$\int_0^1 \tilde{f}(x)^2 dx = \int_0^1 \int_{\mathbb{R}} 2(y - x)(1_\Delta - 1_{\Delta_0})(x, y) dx dy = \int_0^1 \tilde{g}(y)^2 dy.$$

So  $g \in \mathcal{D}^*$  and  $\rho(g) = \rho(f)$ . We may construct a disturbance flow  $\Phi$  with disturbance  $f$  from a sequence  $(\Theta_n : n \in \mathbb{Z})$  of independent random variables, uniformly distributed on  $(0, 1]$ , by

$$\Phi_I^\pm = f_{\Theta_n}^\pm \circ \cdots \circ f_{\Theta_m}^\pm$$

where  $m$  and  $n$  are respectively the minimal and maximal integers in  $\rho I$ . Then

$$\hat{\Phi}_I^\pm = g_{\Theta_{-n}}^\pm \circ \cdots \circ g_{\Theta_{-m}}^\pm.$$

Since  $(\Theta_n : n \in \mathbb{Z})$  and  $(\Theta_{-n} : n \in \mathbb{Z})$  have the same distribution, it follows that  $\hat{\Phi}$  is a disturbance flow with disturbance  $g$ .  $\square$

We get as a corollary the reversibility of the limit, which is already known in various guises. See, for example, [1, 7, 18].

**Corollary 7.2.** *The laws  $\mu_A$  and  $\bar{\mu}_A$  of the coalescing Brownian flow on the circle and the line are invariant under time-reversal.*

*Proof.* Write  $\hat{\mu}_A$  for the image measure of  $\mu_A$  under time reversal and  $\hat{\bar{\mu}}_A$  for the image measure of  $\bar{\mu}_A$ . Fix  $r \in (0, 1/2]$  and define  $f \in \mathcal{D}^*$  by

$$f^+(n+x) = n + (r \vee x \wedge (1-r)), \quad n \in \mathbb{Z}, \quad x \in [0, 1).$$

Then  $\tilde{f}^+(x) = ((r-x) \vee 0) + ((1-r-x) \wedge 0)$  for  $x \in [0, 1)$ , so  $\rho(f) = 3/(2r^3)$  and

$$\int_0^1 \tilde{f}(x) \tilde{f}(x+a) dx = 0, \quad 2r \leq a \leq 1-2r,$$

so  $\lambda(f) \leq 2r$ . Moreover  $\rho(f^{-1}) = \rho(f)$  and  $\lambda(f^{-1}) \leq 2r$ . Consider the limit  $r \rightarrow 0$ . By Theorem 6.1, we know that  $\mu_A^f \rightarrow \mu_A$  and  $\mu_A^{f^{-1}} \rightarrow \mu_A$ , weakly on  $D^\circ(\mathbb{R}, \mathcal{D})$ . Since the time-reversal map  $\phi \mapsto \hat{\phi}$  is an isometry, it follows, using the preceding proposition, that  $\mu_A^{f^{-1}} = \hat{\mu}_A^f \rightarrow \hat{\mu}_A$ , weakly on  $D^\circ(\mathbb{R}, \mathcal{D})$ . Hence  $\mu_A = \hat{\mu}_A$ .

Similarly, we can take  $\varepsilon = \sqrt{r}$  to ensure that  $\varepsilon \rightarrow 0$  and  $\varepsilon^3 \rho(f) \rightarrow \infty$  and  $\lambda(f, \varepsilon) \rightarrow 0$  and  $\lambda(f^{-1}, \varepsilon) \rightarrow 0$ . Then by Theorem 6.1, we have  $\mu_A^{f, \varepsilon} \rightarrow \bar{\mu}_A$  and  $\mu_A^{f^{-1}, \varepsilon} \rightarrow \bar{\mu}_A$ , weakly on  $D^\circ(\mathbb{R}, \bar{\mathcal{D}})$ . But then also  $\mu_A^{f^{-1}, \varepsilon} = \hat{\mu}_A^{f, \varepsilon} \rightarrow \hat{\mu}_A$ , weakly on  $D^\circ(\mathbb{R}, \mathcal{D})$ , so  $\bar{\mu}_A = \hat{\mu}_A$ .  $\square$

From the flow-level result Theorem 6.1 we can deduce weak convergence also for paths running forwards and backwards in time from a given sequence of points  $E = (e_k : k \in \mathbb{N})$  in  $\mathbb{R}^2$ . For  $e = (s, x) \in \mathbb{R}^2$ , define  $\bar{D}_e = \{\xi \in D(\mathbb{R}, \mathbb{R}) : \xi_s = x\}$  and set  $\bar{D}_E = \prod_{k=1}^\infty \bar{D}_{e_k}$ . (The ‘bar’ notation in this discussion is not intended to suggest any relation to objects introduced earlier for the non-periodic case, where a similar notation was also used.) For  $\phi \in D^\circ(\mathbb{R}, \mathcal{D})$ , define

$$\bar{Z}_t^{e, \pm}(\phi) = \begin{cases} \phi_{(s, t]}^\pm(x), & t \geq s, \\ (\phi^{-1})_{(t, s]}^\pm(x), & t < s. \end{cases}$$

Then  $\bar{Z}^{e, \pm}(\phi) \in \bar{D}_e$  and extends  $Z^{e, \pm}(\phi)$ , as defined in Section 5, from  $[s, \infty)$  to the whole of  $\mathbb{R}$ . For all  $e \in \mathbb{R}^2$ , we have  $\bar{Z}^{e, +} = \bar{Z}^{e, -}$  almost everywhere on  $D^\circ(\mathbb{R}, \mathcal{D})$  for both  $\mu_A$  and  $\mu_A^f$ , for any disturbance function  $f$ . So we drop the  $\pm$ . Denote by  $\bar{\mu}_E^f$  the law of  $(\bar{Z}^{e_k} : k \in \mathbb{N})$  on  $\bar{D}_E$  under  $\mu_A^f$  and by  $\bar{\mu}_E$  the corresponding law under  $\mu_A$ .

**Corollary 7.3.** *We have  $\bar{\mu}_E^f \rightarrow \bar{\mu}_E$  weakly on  $\bar{D}_E$ , uniformly in  $f \in \mathcal{D}^*$ , as  $\rho(f) \rightarrow \infty$  and  $\lambda(f) \rightarrow 0$ .*

*Proof.* We can check that  $\bar{Z}^{(s,x),+}$  is continuous as a map  $D^\circ(\mathbb{R}, \mathcal{D}) \rightarrow \bar{D}_{(s,x)}$  at  $\phi \in C^\circ(\mathbb{R}, \mathcal{D})$  provided

$$\bar{Z}^{(s,x\pm\delta),+}(\phi) \rightarrow \bar{Z}^{(s,x),+}(\phi)$$

uniformly on  $\mathbb{R}$  as  $\delta \rightarrow 0$ . Since this property holds for  $\mu_A$  almost all  $\phi$ , the claimed limit follows from Theorem 6.1 by a standard property of weak convergence.  $\square$

Weak convergence of the forwards paths to coalescing Brownian motions was shown in Proposition 2.3. The corresponding backwards property is immediate from the fact that the time reversal of a disturbance flow is another such flow. What is new in the result just proved is the identification of the limit of the joint law of these backwards and forwards paths – which has the property that the bi-infinite paths never cross.

## 8 Appendix

### 8.1 Some properties of the space $\mathcal{D}$ of non-decreasing functions of degree 1

We give proofs in this subsection of a number of assertions made in Section 3.

**Proposition 8.1.** *The map  $f \mapsto f^\times : \mathcal{D} \rightarrow \mathcal{S}$  is a well-defined bijection, with inverse given by*

$$f^-(x) = \inf\{t + f^\times(t) : t \in \mathbb{R}, x = t - f^\times(t)\}, \quad f^+(x) = \sup\{t + f^\times(t) : t \in \mathbb{R}, x = t - f^\times(t)\}.$$

*Proof.* Recall that  $f^\times(t) = t - x$ , where  $x$  is the unique point such that  $f^-(x) \leq 2t - x \leq f^+(x)$ . The periodicity of  $f^\times$  is an easy consequence of the degree 1 condition. We now show that  $f^\times$  is a contraction. Fix  $s, t \in \mathbb{R}$  and suppose that  $f^\times(s) = s - y$ . Switching the roles of  $s$  and  $t$  if necessary, we may assume without loss that  $x \geq y$ . If  $x = y$ , then  $f^\times(s) - f^\times(t) = s - t$ . On the other hand, if  $x > y$ , then  $2s - y \leq f^+(y) \leq f^-(x) \leq 2t - x$ , so

$$-(t - s) \leq -(t - s) + (2t - x) - (2s - y) = f^\times(t) - f^\times(s) = (t - s) - (x - y) < t - s.$$

In both cases, we see that  $|f^\times(t) - f^\times(s)| \leq |t - s|$ . Hence  $f^\times \in \mathcal{S}$ .

Suppose now that  $g \in \mathcal{S}$ . Consider, for each  $x \in \mathbb{R}$ , the set

$$I_x = \{t + g(t) : t \in \mathbb{R}, x = t - g(t)\}.$$

Since  $g$  is a contraction, these sets are all intervals, and, since  $g$  is bounded, they cover  $\mathbb{R}$ . For  $x, y \in \mathbb{R}$  with  $x > y$ , and for  $s, t \in \mathbb{R}$  with  $x = t - g(t), y = s - g(s)$ , we have  $t - s - (g(t) - g(s)) = x - y > 0$ , so  $s \leq t$ , and so

$$t + g(t) - (s + g(s)) = t - s + (g(t) - g(s)) \geq 0.$$

Define  $h^+(y) = \sup I_y$  and  $h^-(x) = \inf I_x$ . We have shown that  $h^+(y) \leq h^-(x)$ . Moreover, since the intervals  $I_x$  cover  $\mathbb{R}$ , the functions  $h^\pm$  must be the left-continuous and right-continuous versions of a non-decreasing function  $h$ , which then has the degree 1 property, because  $g$  is periodic. Thus  $h \in \mathcal{D}$ .

For each  $t \in \mathbb{R}$ , we have  $h^\times(t) = t - x$ , where  $2t - x \in I_x$ , and so  $2t - x = s + g(s)$  for some  $s \in \mathbb{R}$  with  $x = s + g(s)$ . Then  $s = t$  and so  $h^\times(t) = g(t)$ . Hence  $h^\times = g$ . On the other hand, if we take  $g = f^\times$  and if  $x$  is a point of continuity of  $f$ , then we find  $I_x = \{f(x)\}$ , so  $h^+(x) = h^-(x) = f(x)$ . Hence  $h = f$ . We have now shown that  $f \mapsto f^\times : \mathcal{D} \rightarrow \mathcal{S}$  is a bijection, and that its inverse has the claimed form.  $\square$

**Proposition 8.2.** *For  $f, g \in \mathcal{D}$  and  $\varepsilon > 0$ ,*

$$d_{\mathcal{D}}(f, g) \leq \varepsilon \iff f^-(x - \varepsilon) - \varepsilon \leq g^-(x) \leq g^+(x) \leq f^+(x + \varepsilon) + \varepsilon \text{ for all } x \in \mathbb{R}.$$

*Moreover, for any sequence  $(f_n : n \in \mathbb{N})$  in  $\mathcal{D}$ ,*

$$f_n \rightarrow f \text{ in } \mathcal{D} \iff f_n^+(x) \rightarrow f(x) \text{ at all points } x \in \mathbb{R} \text{ where } f \text{ is continuous.}$$

*Proof.* Suppose that  $d_{\mathcal{D}}(f, g) \leq \varepsilon$  and that  $x$  is a continuity point of  $g$ . Then  $g(x) = t + g^\times(t)$  for some  $t \in \mathbb{R}$  with  $x = t - g^\times(t)$ . We must have  $x + \varepsilon \geq t - f^\times(t)$  and  $g(x) \leq t + f^\times(t) + \varepsilon$ , so  $f^+(x + \varepsilon) + \varepsilon \geq t + f^\times(t) + \varepsilon \geq g(x)$ . Similarly  $f^-(x - \varepsilon) - \varepsilon \leq g(x)$ . These inequalities extend to all  $x \in \mathbb{R}$  by taking left and right limits along continuity points.

Conversely, suppose that  $t \in \mathbb{R}$  is such that  $|f^\times(t) - g^\times(t)| = d_{\mathcal{D}}(f, g)$  and let  $x = t - g^\times(t)$  and  $y = t - f^\times(t)$ . Then  $x$  is the unique point with  $g^-(x) + x \leq 2t \leq g^+(x) + x$  and  $y$  is the unique point such that  $f^-(y) + y \leq 2t \leq f^+(y) + y$ . Hence  $f^-(x - \varepsilon) - \varepsilon \leq g^-(x) \leq g^+(x) \leq f^+(x + \varepsilon) + \varepsilon$  implies  $y \in [x - \varepsilon, x + \varepsilon]$  and so  $d_{\mathcal{D}}(f, g) = |y - x| \leq \varepsilon$ .

It follows directly that for any sequence  $(f_n : n \in \mathbb{N})$  in  $\mathcal{D}$ , if  $d_{\mathcal{D}}(f_n, f) \rightarrow 0$  as  $n \rightarrow \infty$ , then  $f_n^+(x) \rightarrow f(x)$  at all points  $x \in \mathbb{R}$  where  $f$  is continuous.

Now suppose  $f_n^+(x) \rightarrow f(x)$  at all points  $x \in \mathbb{R}$  where  $f$  is continuous. By equicontinuity, it will suffice to show that  $f_n^\times(t) \rightarrow f^\times(t)$  for each  $t \in \mathbb{R}$ . Set  $x = t - f^\times(t)$  and  $x_n = t - f_n^\times(t)$ . Given  $\varepsilon > 0$ , choose  $y_1 \in (x - \varepsilon, x)$  and  $y_2 \in (x, x + \varepsilon)$ , both points of continuity of  $f$ . Now  $f(y_1) + y_1 < 2t < f(y_2) + y_2$ , so there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ , we have  $f_n^+(y_1) + y_1 < 2t < f_n^+(y_2) + y_2$ , which implies  $x_n \in [y_1, y_2]$  and hence  $|f_n^\times(t) - f^\times(t)| < \varepsilon$ , as required.  $\square$

**Proposition 8.3.** *Suppose  $f_n \rightarrow f, g_n \rightarrow g, h_n \rightarrow h$  in  $\mathcal{D}$  with  $h_n^+ \leq f_n^+ \circ g_n^+$  for all  $n$ . Then  $h^+ \leq f^+ \circ g^+$ .*

*Proof.* It will suffice to establish the inequality at all points  $x$  where  $g$  and  $h$  are both continuous. Given  $\varepsilon > 0$ , since  $f^+$  is right-continuous, there exists a point  $y > g(x)$  where  $f$  is continuous and such that  $f(y) < f^+(g(x)) + \varepsilon$ . Then  $f_n^+(y) < f^+(g(x)) + \varepsilon$  and  $g_n^+(x) \leq y$  eventually, so

$$h_n^+(x) \leq f_n^+(g_n^+(x)) \leq f_n^+(y) < f^+(g(x)) + \varepsilon$$

eventually. Hence  $h^+(x) = \lim_{n \rightarrow \infty} h_n^+(x) \leq f^+(g(x))$ , as required.  $\square$

## 8.2 Some properties of the continuous flow-space $C^\circ(\mathbb{R}, \mathcal{D})$ and cadlag flow-space $D^\circ(\mathbb{R}, \mathcal{D})$

We give proofs in this subsection of a number of assertions made in Sections 3 and 5.

**Proposition 8.4.** *For  $(s, x) \in \mathbb{R}^2$  and  $\phi \in D^\circ(\mathbb{R}, \mathcal{D})$ , the map*

$$t \mapsto \phi_{(s,t]}^+(x) : [s, \infty) \rightarrow \mathbb{R}$$

*is cadlag, and is moreover continuous whenever  $\phi \in C^\circ(\mathbb{R}, \mathcal{D})$ .*

*Proof.* Given  $t \geq s$  and  $\varepsilon > 0$ , we can choose  $\delta > 0$  so that  $d_{\mathcal{D}}(\phi_{(t,u]}, \text{id}) < \varepsilon/2$  for all  $u \in (t, t + \delta]$ . For such  $u$  and for  $x$  a point of continuity of  $\phi_{(s,t]}$ , we have

$$\phi_{(s,t]}^+(x) - \varepsilon = \phi_{(s,t]}^-(x) - \varepsilon \leq \phi_{(t,u]}^-(x) - \varepsilon \leq \phi_{(t,u]}^-(x) \circ \phi_{(s,t]}^-(x) \leq \phi_{(s,u]}^-(x) \leq \phi_{(s,u]}^+(x) \leq \phi_{(t,u]}^+(x) \circ \phi_{(s,t]}^+(x) \leq \phi_{(s,t]}^+(x) + \varepsilon,$$

so  $|\phi_{(s,u]}^+(x) - \phi_{(s,t]}^+(x)| \leq \varepsilon$ . The final estimate extends to all  $x$  by right-continuity. Hence the map is right continuous. A similar argument shows that, for  $u \in (s, t)$ , we have  $|\phi_{(s,u]}^+(x) - \phi_{(s,t)}^+(x)| \rightarrow 0$  as  $u \rightarrow t$ , so that the map has a left limit at  $t$  given by  $\phi_{(s,t)}^+(x)$ . Finally, if  $\phi \in C^\circ(\mathbb{R}, \mathcal{D})$ , then  $\phi_{(s,t)} = \phi_{(s,t]}$ , so the map is continuous.  $\square$

**Proposition 8.5.** *For all  $\phi \in C^\circ(\mathbb{R}, \mathcal{D})$ , the map  $(s, t) \mapsto \phi_{ts} : \{(s, t) : s \leq t\} \rightarrow \mathcal{D}$  is continuous. Moreover, for all  $\phi \in D^\circ(\mathbb{R}, \mathcal{D})$  and for any sequence of bounded intervals  $I_n \rightarrow I$ , we have  $\phi_{I_n} \rightarrow \phi_I$ .*

*Proof.* The first assertion follows from the second: given  $\phi \in C^\circ(\mathbb{R}, \mathcal{D})$  and sequences  $s_n \rightarrow s$  and  $t_n \rightarrow t$ , then, passing to a subsequence if necessary, we can assume that  $(s_n, t_n] \rightarrow I$  for some interval  $I$  with  $\inf I = s$  and  $\sup I = t$ . Then, by the second assertion, we have  $\phi_{t_n s_n} \rightarrow \phi_I = \phi_{ts}$ , as required.

So, let us fix  $\phi \in D^\circ(\mathbb{R}, \mathcal{D})$  and a sequence of bounded intervals  $I_n \rightarrow I$ . By combining the cadlag and weak flow properties, we can show the following variant of the cadlag property: for all  $t \in \mathbb{R}$ , we have

$$\phi_{[s,t)} \rightarrow \text{id} \quad \text{as } s \uparrow t, \quad \phi_{(t,u]} \rightarrow \text{id} \quad \text{as } u \downarrow t. \quad (13)$$

For each  $n$ , there exist two disjoint intervals  $J_n$  and  $J'_n$ , possibly empty, such that  $I \Delta I_n = J_n \cup J'_n$ . For any such  $J_n$  and  $J'_n$ , using the weak flow property, we obtain

$$d_{\mathcal{D}}(\phi_I, \phi_{I_n}) \leq \|\phi_{J_n} - \text{id}\| + \|\phi_{J'_n} - \text{id}\|.$$

Set  $s = \inf I$ ,  $s_n = \inf I_n$ ,  $t = \sup I$  and  $t_n = \sup I_n$ . Then  $s_n \rightarrow s$ ,  $t_n \rightarrow t$ , and

if  $s \in I$  then  $s \in I_n$  eventually,    if  $s \notin I$  then  $s \notin I_n$  eventually,

if  $t \in I$  then  $t \in I_n$  eventually,    if  $t \notin I$  then  $t \notin I_n$  eventually.

Hence, using the cadlag property or (13), or both, we find that  $\phi_{J_n} \rightarrow \text{id}$  and  $\phi_{J'_n} \rightarrow \text{id}$ , which proves the proposition.  $\square$

**Proposition 8.6.** *The metrics  $d_C$  and  $d_D$  generate the same topology on  $C^\circ(\mathbb{R}, \mathcal{D})$ .*

*Proof.* On comparing the definitions of  $d_n^C$  and  $d_n^D$  for each  $n \in \mathbb{N}$ , and considering the choice  $\lambda = \text{id}$ , we see that  $d_D \leq d_C$ . Hence, it will suffice to show, given  $\phi \in C^\circ(\mathbb{R}, \mathcal{D})$ ,  $n \in \mathbb{N}$  and  $\varepsilon > 0$ , that there exists  $\varepsilon' > 0$  such that, for all  $\psi \in C^\circ(\mathbb{R}, \mathcal{D})$ , we have  $d_C^{(n)}(\phi, \psi) < \varepsilon$  whenever  $d_{n+1}^D(\phi, \psi) < \varepsilon'$ . By the preceding proposition, there exists a  $\delta \in (0, 1]$  such that  $d_{\mathcal{D}}(\phi_{ts}, \phi_{t's'}) < \varepsilon/2$  whenever  $|s - s'|, |t - t'| \leq \delta$  and  $s, t \in (-n, n)$ . Set  $\varepsilon' = \delta \wedge (\varepsilon/2)$  and suppose that  $d_{n+1}^D(\phi, \psi) < \varepsilon'$ . Then there exists an increasing homeomorphism  $\lambda$  of  $\mathbb{R}$ , with  $|\lambda(t) - t| \leq \delta$  for all  $t$ , such that, for all intervals  $I$ , we have  $\|\chi_{n+1}(I)\psi_I^\times - \chi_{n+1}(\lambda(I))\phi_{\lambda(I)}^\times\| < \varepsilon/2$ . Given  $s, t \in (-n, n)$  with  $s < t$ , take  $I = (s, t]$ . Then  $\chi_{n+1}(I) = \chi_{n+1}(\lambda(I)) = 1$ , so  $d_{\mathcal{D}}(\phi_{\lambda(t)\lambda(s)}, \psi_{ts}) = \|\psi_I^\times - \phi_{\lambda(I)}^\times\| < \varepsilon/2$ . But then, for all such  $s, t$ , we have

$$d_{\mathcal{D}}(\phi_{ts}, \psi_{ts}) \leq d_{\mathcal{D}}(\phi_{ts}, \phi_{\lambda(t)\lambda(s)}) + d_{\mathcal{D}}(\phi_{\lambda(t)\lambda(s)}, \psi_{ts}) < \varepsilon,$$

so  $d_C^{(n)}(\phi, \psi) < \varepsilon$ , as required.  $\square$

**Proposition 8.7.** *The metric spaces  $(C^\circ(\mathbb{R}, \mathcal{D}), d_C)$  and  $(D^\circ(\mathbb{R}, \mathcal{D}), d_D)$  are complete and separable.*

*Proof.* The argument for completeness is a variant of the corresponding argument for the usual Skorokhod space  $D(\mathbb{R}, S)$  of cadlag paths in complete separable metric space  $S$ , as found for example in [2]. Suppose then that  $(\psi^n)_{n \geq 1}$  is a Cauchy sequence in  $D^\circ(\mathbb{R}, \mathcal{D})$ . There exists



a subsequence  $\phi^k = \psi^{n_k}$  such that  $d_D^{(n)}(\phi^n, \phi^{n+1}) < 2^{-n}$  for all  $n \geq 1$ . It will suffice to find a limit in  $D^\circ(\mathbb{R}, \mathcal{D})$  for  $(\phi^n)_{n \geq 1}$ . There exist increasing homeomorphisms  $\mu_n$  of  $\mathbb{R}$  for which  $\gamma(\mu_n) < 2^{-n}$  and

$$d_{\mathcal{D}}(\phi_I^n, \phi_{\mu_n(I)}^{n+1}) < 2^{-n}, \quad I \cup \mu_n(I) \subseteq (-n, n).$$

For each  $n \geq 1$ , the sequence  $(\mu_{n+m} \circ \cdots \circ \mu_n)_{m \geq 1}$  converges uniformly on  $\mathbb{R}$  to an increasing homeomorphism,  $\lambda_n$  say, with  $\gamma(\lambda_n) < 2^{-n+1}$ . Then  $\mu_n \circ \lambda_n^{-1} = \lambda_{n+1}^{-1}$ , so

$$d_{\mathcal{D}}(\phi_{\lambda_n^{-1}(I)}^n, \phi_{\lambda_{n+1}^{-1}(I)}^{n+1}) < 2^{-n}, \quad I \subseteq (-n+1, n-1).$$

So, for all  $m \geq n$ ,

$$d_{\mathcal{D}}(\phi_{\lambda_n^{-1}(I)}^n, \phi_{\lambda_{n+m}^{-1}(I)}^{n+m}) < 2^{-n+1}, \quad I \subseteq (-n+1, n-1). \quad (14)$$

Hence, for all bounded intervals  $I \subseteq \mathbb{R}$ ,  $(\phi_{\lambda_n^{-1}(I)}^n)_{n \geq 1}$  is a Cauchy sequence in  $\mathcal{D}$ , which, since  $\mathcal{D}$  is complete, has a limit  $\phi_I \in \mathcal{D}$ . On letting  $m \rightarrow \infty$  in (14), we obtain

$$d_{\mathcal{D}}(\phi_{\lambda_n^{-1}(I)}^n, \phi_I) < 2^{-n+1}, \quad I \subseteq (-n+1, n-1).$$

By Proposition 8.3,  $\phi = (\phi_I : I \subseteq \mathbb{R})$  has the weak flow property. To see that  $\phi$  is cadlag, suppose given  $\varepsilon > 0$  and  $t \in \mathbb{R}$ . Choose  $n$  such that  $2^{-n+1} \leq \varepsilon/3$  and  $|t| \leq n-2$ . Then choose  $\delta \in (0, 1]$  such that

$$d_{\mathcal{D}}(\phi_{\lambda_n^{-1}(s,t)}^n, \text{id}) < \varepsilon/3, \quad d_{\mathcal{D}}(\phi_{\lambda_n^{-1}(t,u)}^n, \text{id}) < \varepsilon/3$$

whenever  $s \in (t - \delta, t)$  and  $u \in (t, t + \delta)$ . For such  $s$  and  $u$ , we then have

$$d_{\mathcal{D}}(\phi_{(s,t)}, \text{id}) < \varepsilon, \quad d_{\mathcal{D}}(\phi_{(t,u)}, \text{id}) < \varepsilon.$$

Hence  $\phi \in D^\circ(\mathbb{R}, \mathcal{D})$ . For  $m \leq n-3$ , we have

$$\begin{aligned} d_D^{(m)}(\phi^n, \phi) &\leq \gamma(\lambda_n) \vee \sup_{I \subseteq (-m-2, m+2)} \|\chi_m(\lambda_n^{-1}(I))\phi_{\lambda_n^{-1}(I)}^{n \times} - \chi_m(I)\phi_I^\times\| \\ &\leq \gamma(\lambda_n) \vee \sup_{I \subseteq (-m-2, m+2)} \left\{ d_{\mathcal{D}}(\phi_{\lambda_n^{-1}(I)}^n, \phi_I) + \gamma(\lambda_n) \|\phi_I^\times\| \right\} \\ &\leq 2^{-n+1} (1 + \sup_{I \subseteq (-m-2, m+2)} \|\phi_I^\times\|). \end{aligned}$$

Hence  $d_D(\phi^n, \phi) \rightarrow 0$  as  $n \rightarrow \infty$ . We have shown that  $D^\circ(\mathbb{R}, \mathcal{D})$  is complete. If the sequence  $(\phi^n)_{n \geq 1}$  in fact lies in  $C^\circ(\mathbb{R}, \mathcal{D})$ , then, by an obvious variation of the argument for the cadlag property, the limit  $\phi$  also lies in  $C^\circ(\mathbb{R}, \mathcal{D})$ . Hence  $C^\circ(\mathbb{R}, \mathcal{D})$  is also complete. In particular,  $C^\circ(\mathbb{R}, \mathcal{D})$  is a closed subspace in  $D^\circ(\mathbb{R}, \mathcal{D})$ .

We turn to the question of separability. Let us write  $D_N$  for the set of those  $\phi \in D^\circ(\mathbb{R}, \mathcal{D})$  such that:

- (i) for some  $n \in \mathbb{N}$  and some rationals  $t_1 < \dots < t_n$ , we have  $\phi_J = \text{id}$  for all time intervals  $J$  which do not intersect the set  $\{t_1, \dots, t_n\}$ ;
- (ii) for all other time intervals  $I$ , the maps  $\phi_I$  and  $\phi_I^{-1}$  on  $\mathbb{R}$  are constant on all space intervals which do not intersect  $2^{-N}\mathbb{Z}$ .

Note that each  $\phi \in D_N$  is determined by the maps  $\phi_{(t_k, t_m]}$ , for integers  $0 \leq k < m \leq n$ , where  $t_0 < t_1$ , and for each of these maps there are only countably many possibilities (finitely many if we insist that  $\phi(0) \in [0, 1)$ ). Hence  $D_N$  is countable and so is  $D_* = \bigcup_{N \geq 1} D_N$ . We shall show that  $D_*$  is also dense in  $D^\circ(\mathbb{R}, \mathcal{D})$ .

Fix  $\phi \in D^\circ(\mathbb{R}, \mathcal{D})$  and  $n_0 \geq 1$ . It will suffice to find, for a given  $\varepsilon > 0$ , a  $\psi \in D_*$  with  $d_D^{(n_0)}(\phi, \psi) < \varepsilon$ . By the cadlag property and compactness, there exist  $n \in \mathbb{N}$  and reals  $s_1 < \dots < s_n$  in  $I_0 = (-n_0 - 1, n_0 + 1)$  such that  $d_{\mathcal{D}}(\phi_I, \text{id}) < \varepsilon/4$  for every subinterval  $I$  of  $I_0$  which does not intersect  $\{s_1, \dots, s_n\}$ . Then we can find rationals  $t_1 < \dots < t_n$  in  $I_0$  and an increasing homeomorphism  $\lambda$  of  $\mathbb{R}$ , with  $\lambda(t) = t$  for  $t \notin I_0$ , with  $\gamma(\lambda) \sup_{I \subseteq I_0} \|\phi_I^\times\| < \varepsilon/4$ , and such that  $\lambda(t_m) = s_m$  for all  $m$ . Set  $s_0 = t_0 = -n_0 - 1$ .

For  $f \in \mathcal{D}$ , write  $\Delta(f)$  for the set of points where  $f$  is not continuous. Define, for  $m = 0, 1, \dots, n$ ,

$$\Delta_m = \bigcup_{k=0}^{m-1} \Delta(\phi_{(s_k, s_m]}^{-1}) \cup \bigcup_{k=m+1}^n \Delta(\phi_{(s_m, s_k]}).$$

Then  $\Delta_m$  is countable, so we can choose  $N \geq 1$  with  $16 \cdot 2^{-N} \leq \varepsilon$  and choose  $\varepsilon_m \in \mathbb{R}$  with  $|\varepsilon_m| \leq 2^{-N}$  such that

$$\tau_m(\Delta_m) \cap 2^{-N}\mathbb{Z} = \emptyset, \quad m = 0, 1, \dots, n,$$

where  $\tau_m(x) = x + \varepsilon_m$ . Set

$$\delta^-(x) = 2^N \lceil 2^{-N} x \rceil, \quad \delta^+(x) = 2^N \lfloor 2^{-N} x \rfloor + 1.$$

Note that  $\delta = \{\delta^-, \delta^+\} \in \mathcal{D}$ . Define for  $0 \leq k < m \leq n$

$$\psi_{(t_k, t_m]}^- = (\delta^{-1})^- \circ (\tau_m)^{-1} \circ \phi_{(s_k, s_m]}^- \circ \tau_k \circ \delta^-, \quad \psi_{(t_k, t_m]}^+ = (\delta^{-1})^+ \circ (\tau_m)^{-1} \circ \phi_{(s_k, s_m]}^+ \circ \tau_k \circ \delta^+.$$

Then  $\psi_{(t_k, t_m]} = \{\psi_{(t_k, t_m]}^-, \psi_{(t_k, t_m]}^+\} \in \mathcal{D}$  by our choice of  $\varepsilon_k$  and  $\varepsilon_m$ . Moreover  $\delta^+ \circ (\delta^{-1})^+ \geq \text{id}$  and  $\delta^- \circ (\delta^{-1})^- \leq \text{id}$  so, for  $0 \leq m < m' < m'' \leq n$ , we obtain the inequalities

$$\psi_{(t_{m'}, t_{m''}]}^- \circ \psi_{(t_m, t_{m'}]}^- \leq \psi_{(t_m, t_{m''}]}^- \leq \psi_{(t_m, t_{m''}]}^+ \leq \psi_{(t_{m'}, t_{m''}]}^+ \circ \psi_{(t_m, t_{m'}]}^+$$

from the corresponding inequalities for  $\phi$ . We use the equations  $\|\delta - \text{id}\| = 2^{-N}$  and  $\|\tau_m - \text{id}\| = |\varepsilon_m|$  to see that

$$d_{\mathcal{D}}(\phi_{(s_k, s_m]}, \psi_{(t_k, t_m]}) \leq 4 \cdot 2^{-N}, \quad 0 \leq k < m \leq n.$$

Define  $\psi_J = \psi_{(t_k, t_m]}$  for all intervals  $J$  such that  $J \cap \{t_1, \dots, t_n\} = \{t_{k+1}, \dots, t_m\}$ . For such intervals  $J$ , with  $J \subseteq I_0$ , we have  $d_{\mathcal{D}}(\phi_{(s_k, s_m] \setminus \lambda(J)}, \text{id}) < \varepsilon/4$  and  $d_{\mathcal{D}}(\phi_{\lambda(J) \setminus (s_k, s_m]}, \text{id}) < \varepsilon/4$  so, using the weak flow property for  $\phi$ ,

$$d_{\mathcal{D}}(\psi_J, \phi_{\lambda(J)}) \leq d_{\mathcal{D}}(\psi_{(t_k, t_m]}, \phi_{(s_k, s_m]}) + d_{\mathcal{D}}(\phi_{(s_k, s_m]}, \phi_{\lambda(J)}) \leq 4 \cdot 2^{-N} + 2\varepsilon/4 < 3\varepsilon/4.$$

Define  $\psi_J = \text{id}$  for all intervals  $J$  which do not intersect  $\{t_1, \dots, t_n\}$ . For such intervals  $J$  with  $J \subseteq I_0$ , we have  $d_{\mathcal{D}}(\psi_J, \phi_{\lambda(J)}) \leq d_{\mathcal{D}}(\text{id}, \phi_{\lambda(J)}) \leq \varepsilon/4$ . Now  $\psi \in D_N$  and

$$d_D^{(n_0)}(\phi, \psi) \leq \gamma(\lambda) \vee \sup_{J \subseteq I_0} \{d_{\mathcal{D}}(\psi_J, \phi_{\lambda(J)}) + \gamma(\lambda) \|\phi_J^\times\|\} < \varepsilon,$$

as required. This proves that  $D^\circ(\mathbb{R}, \mathcal{D})$  is separable and, since  $C^\circ(\mathbb{R}, \mathcal{D})$  is a closed subspace of  $D^\circ(\mathbb{R}, \mathcal{D})$ , it follows that  $C^\circ(\mathbb{R}, \mathcal{D})$  is also separable.  $\square$

**Proposition 8.8.** *For all  $s, t \in \mathbb{R}$  with  $s < t$ , and all  $x \in \mathbb{R}$ , the map  $\phi \mapsto \phi_{ts}^+(x)$  on  $C^\circ(\mathbb{R}, \mathcal{D})$  is Borel measurable. Moreover the Borel  $\sigma$ -algebra on  $C^\circ(\mathbb{R}, \mathcal{D})$  is generated by the set of all such maps with  $s, t$  and  $x$  rational.*

*For all bounded intervals  $I \subseteq \mathbb{R}$  and all  $x \in \mathbb{R}$ , the map  $\phi \mapsto \phi_I^+(x)$  on  $D^\circ(\mathbb{R}, \mathcal{D})$  is Borel measurable. Moreover the Borel  $\sigma$ -algebra on  $D^\circ(\mathbb{R}, \mathcal{D})$  is generated by the set of all such maps with  $I = (s, t]$  and with  $s, t$  and  $x$  rational.*

*Proof.* The assertions for  $C^\circ(\mathbb{R}, \mathcal{D})$  can be proved more simply than those for  $D^\circ(\mathbb{R}, \mathcal{D})$ . We omit details of the former, but note that these follow also from the latter, by general measure theoretic arguments, given what we already know about the two spaces.

The proof for  $D^\circ(\mathbb{R}, \mathcal{D})$  is an adaptation of the analogous result for the classical Skorokhod space, see for example [11, page 335]. We prove first the Borel measurability of the evaluation maps. Given a bounded interval  $I$  and  $x \in \mathbb{R}$ , we can find  $s_n, t_n \in \mathbb{R}$  such that  $(s_n, t_n] \rightarrow I$  as  $n \rightarrow \infty$ . Then  $\phi_I^+(x) = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \phi_{(s_n, t_n]}^+(x + 1/m)$ , by Proposition 8.5. Hence, it will suffice to consider intervals  $I$  of the form  $(s, t]$ . Fix  $s, t$  and  $x$  and define for each  $m, n \in \mathbb{N}$  a function  $F_{m,n}$  on  $D^\circ(\mathbb{R}, \mathcal{D})$  by

$$F_{m,n}(\phi) = \int_s^{s+1/n} \int_t^{t+1/n} \int_x^{x+1/m} \phi_{(s', t']}^+(x') dx' dt' ds'.$$

Suppose  $\phi^k \rightarrow \phi$  in  $D^\circ(\mathbb{R}, \mathcal{D})$ . We can choose increasing homeomorphisms  $\lambda_k$  of  $\mathbb{R}$  such that,  $\gamma(\lambda_k) \rightarrow 0$  and, uniformly in  $r \in [s-1, s+1]$  and  $u \in [t-1, t+1]$ , we have

$$d_{\mathcal{D}}(\phi_{\lambda_k(r, u]}^k, \phi_{(r, u]}) \rightarrow 0.$$

Define

$$f(r, u) = \int_x^{x+1/m} \phi_{\lambda(r, u]}(x') dx', \quad f_k(r, u) = \int_x^{x+1/m} \phi_{\lambda_k(r, u]}^k(x') dx'.$$

Then  $f_k(r, u) \rightarrow f(r, u)$ , uniformly in  $r \in [s-1, s+1]$  and  $u \in [t-1, t+1]$ . Set  $\mu_k = \lambda_k^{-1}$ . Then

$$\begin{aligned} F_{m,n}(\phi^k) &= \int_{\mu_k(s)}^{\mu_k(s+1/n)} \int_{\mu_k(t)}^{\mu_k(t+1/n)} f_k(r, u) d\lambda_k(u) d\lambda_k(r) \\ &\rightarrow \int_s^{s+1/n} \int_t^{t+1/n} f(r, u) du dr = F_{m,n}(\phi), \end{aligned}$$

so  $F_{m,n}$  is continuous on  $D^\circ(\mathbb{R}, \mathcal{D})$ . By Proposition 8.5, we have

$$\phi_{(s,t]}^+(x) = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{mn^2} F_{m,n}(\phi).$$

Hence  $\phi \mapsto \phi_{(s,t]}^+(x)$  is Borel measurable, as required.

Write now  $\mathcal{E}$  for the  $\sigma$ -algebra on  $D^\circ(\mathbb{R}, \mathcal{D})$  generated by all maps of this form with  $s, t$  and  $x$  rational. It remains to show that  $\mathcal{E}$  contains the Borel  $\sigma$ -algebra of  $D^\circ(\mathbb{R}, \mathcal{D})$ . Write  $\{(I_k, z_k) : k \in \mathbb{N}\}$  for an enumeration of the set  $\{(s, t] : s, t \in \mathbb{Q}, s < t\} \times \mathbb{Q}$ . It is straightforward to show that, for all  $k$ , the map  $\phi \mapsto \phi_{I_k}^\times(z_k)$  is  $\mathcal{E}$ -measurable. Fix  $n \in \mathbb{N}$ ,  $\phi^0 \in D^\circ(\mathbb{R}, \mathcal{D})$ ,  $r \in (0, \infty)$  and  $k \in \mathbb{N}$ , and consider the set

$$A(k, r) = \{\phi \in D^\circ(\mathbb{R}, \mathcal{D}) : (\chi_n(I_1)\phi_{I_1}^\times(z_1), \dots, \chi_n(I_k)\phi_{I_k}^\times(z_k)) \in B(k, r)\},$$

where

$$B(k, r) = \bigcup_{\lambda} \{(y_1, \dots, y_k) \in \mathbb{R}^k : \max_{j \leq k} |y_j - \chi_n(\lambda(I_j))\phi_{\lambda(I_j)}^{0 \times}(z_j)| < r\},$$

where the union is taken over all increasing homeomorphisms  $\lambda$  of  $\mathbb{R}$  with  $\gamma(\lambda) < r$ . Note that  $B(k, r)$  is an open set in  $\mathbb{R}^k$ , so  $A(k, r) \in \mathcal{E}$ , so  $A = \bigcup_{m \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} A(k, r - 1/m) \in \mathcal{E}$ .

Consider the set

$$C = \{\phi \in D^\circ(\mathbb{R}, \mathcal{D}) : d_D^{(n)}(\phi, \phi^0) < r\}.$$

It is straightforward to check from the definition of  $d_D^{(n)}$ , that  $C \subseteq A$ . Suppose that  $\phi \in A$ . We shall show that  $\phi \in C$ . Then  $C = A$ , so  $C \in \mathcal{E}$ , and since sets of this form generate the Borel  $\sigma$ -algebra, we are done.

We can find an  $m \in \mathbb{N}$  and, for each  $k \in \mathbb{N}$ , a  $\lambda_k$  with  $\gamma(\lambda_k) < r - 1/m$  such that

$$|\chi_n(I_j)\phi_{I_j}^\times(z_j) - \chi_n(\lambda_k(I_j))\phi_{\lambda_k(I_j)}^{0 \times}(z_j)| < r - 1/m, \quad j = 1, \dots, k.$$

Without loss of generality, we may assume that the sequence  $(\lambda_k : k \in \mathbb{N})$  converges uniformly on compacts, and that its limit,  $\lambda$  say, satisfies  $\gamma(\lambda) \leq r - 1/m$ . By Proposition 8.5, for each  $j$ , there is an interval  $\hat{I}_j$ , having the same endpoints as  $I_j$  such that  $\phi_{\lambda(\hat{I}_j)}$  is a limit point in  $\mathcal{D}$

of the sequence  $(\phi_{\lambda_k(I_j)} : k \in \mathbb{N})$ , so  $\phi_{\lambda(I_j)}^\times$  is a limit point in  $\mathcal{S}$  of the sequence  $(\phi_{\lambda_k(I_j)}^\times : k \in \mathbb{N})$ . Then

$$|\chi_n(I_j)\phi_{I_j}^\times(z_j) - \chi_n(\lambda(\hat{I}_j))\phi_{\lambda(\hat{I}_j)}^{0\times}(z_j)| \leq r - 1/m,$$

for all  $j$ . For all bounded intervals  $I$  and all  $z \in \mathbb{R}$ , we can find a sequence  $(j_p : p \in \mathbb{N})$  such that  $I_{j_p} \rightarrow I$ ,  $\hat{I}_{j_p} \rightarrow I$  and  $z_{j_p} \rightarrow z$ . So we obtain

$$|\chi_n(I)\phi_I^\times(z) - \chi_n(\lambda(I))\phi_{\lambda(I)}^{0\times}(z)| \leq r - 1/m.$$

Hence  $d_D^{(n)}(\phi, \phi^0) \leq r - 1/m$  and  $\phi \in C$ , as we claimed.  $\square$

Recall that, for  $e = (s, x) \in \mathbb{R}^2$  and  $\phi \in D^\circ(\mathbb{R}, \mathcal{D})$ , we set

$$Z^{e,\pm}(\phi) = (\phi_{(s,t]}^\pm(x) : t \geq s)$$

and for sequences  $E = (e_k : k \in \mathbb{N})$  in  $\mathbb{R}^2$ , we set  $Z^{E,\pm} = (Z^{e_k,\pm} : k \in \mathbb{N})$ . Also

$$C_E^{\circ,\pm} = \{Z^{E,\pm}(\phi) : \phi \in C^\circ(\mathbb{R}, \mathcal{D})\}, \quad D_E^{\circ,\pm} = \{Z^{E,\pm}(\phi) : \phi \in D^\circ(\mathbb{R}, \mathcal{D})\}$$

and

$$D^\circ(E) = \{\phi \in D^\circ(\mathbb{R}, \mathcal{D}) : Z^{E,+}(\phi) = Z^{E,-}(\phi)\}, \quad D_E^\circ = \{Z^E(\phi) : \phi \in D^\circ(E)\}.$$

**Proposition 8.9.** *Let  $E$  be a countable subset of  $\mathbb{R}^2$  containing  $\mathbb{Q}^2$ . Then  $Z^{E,+} : C^\circ(\mathbb{R}, \mathcal{D}) \rightarrow C_E^{\circ,+}$  is a bijection,  $C_E^{\circ,+}$  is a measurable subset of  $C_E$ , and the inverse bijection  $\Phi^{E,+} : C_E^{\circ,+} \rightarrow C^\circ(\mathbb{R}, \mathcal{D})$  is a measurable map. Moreover  $Z^{E,+} : D^\circ(\mathbb{R}, \mathcal{D}) \rightarrow D_E^{\circ,+}$  is also a bijection,  $D_E^{\circ,+}$  is a measurable subset of  $D_E$  and the inverse bijection  $\Phi^{E,+} : D_E^{\circ,+} \rightarrow D^\circ(\mathbb{R}, \mathcal{D})$  is also a measurable map. Moreover the same statements hold with  $+$  replaced by  $-$ , we have  $D_E^\circ = D_E^{\circ,+} \cap D_E^{\circ,-}$  and  $\Phi^{E,+} = \Phi^{E,-}$  on  $D_E^\circ$ .*

*Proof.* We discuss only the cadlag case. The same comments apply as in the preceding proof about the relationship of the cadlag and continuous cases. It is straightforward to see from the density of  $\mathbb{Q}^2$  in  $\mathbb{R}^2$  and the continuity properties of cadlag weak flows that  $Z^{E,+}$  and  $Z^{E,-}$  are both injective on  $D^\circ(\mathbb{R}, \mathcal{D})$ . We shall instead give an explicit description of the ranges  $D_E^{\circ,\pm}$  and explicit constructions of inverse maps  $\Phi^{E,+}$  and  $\Phi^{E,-}$ , which agree on  $D_E^{\circ,+} \cap D_E^{\circ,-}$ , allowing us to establish measurability (as well as injectivity). Consider for  $z \in D_E$  the conditions

$$z_t^{(s,x+n)} = z_t^{(s,x)} + n, \quad s, t, x \in \mathbb{Q}, \quad s < t, \quad n \in \mathbb{Z} \quad (15)$$

and

$$z_t^{(s,x)} = \inf_{y \in \mathbb{Q}, y > x} z_t^{(s,y)}, \quad (s, x) \in E, \quad t \in \mathbb{Q}, \quad t > s. \quad (16)$$

Under these conditions, define for  $s, t \in \mathbb{Q}$  with  $s < t$  and for  $x \in \mathbb{R}$ ,

$$\Phi_{(s,t]}^-(x) = \sup_{y \in \mathbb{Q}, y < x} z_t^{(s,y)}, \quad \Phi_{(s,t]}^+(x) = \inf_{y \in \mathbb{Q}, y > x} z_t^{(s,y)}.$$

Then  $\Phi_{(s,t]} = \{\Phi_{(s,t]}^-, \Phi_{(s,t]}^+\} \in \mathcal{D}$  and

$$\Phi_{(s,t]}^+(x) = z_t^{(s,x)}, \quad s, t, x \in \mathbb{Q}, \quad s < t.$$

Now consider the following additional conditions on  $z$ :

$$\Phi_{(t,u]}^- \circ \Phi_{(s,t]}^- \leq \Phi_{(s,u]}^- \leq \Phi_{(s,u]}^+ \leq \Phi_{(t,u]}^+ \circ \Phi_{(s,t]}^+, \quad s, t, u \in \mathbb{Q}, \quad s < t < u \quad (17)$$

and

for all  $\varepsilon > 0$  and all  $n \in \mathbb{N}$ , there exist  $\delta > 0$ ,  $m \in \mathbb{Z}^+$  and  $u_1, \dots, u_m \in (-n, n)$  such that

$$\|\Phi_{(s,t]} - \text{id}\| < \varepsilon \quad (18)$$

whenever  $s, t \in \mathbb{Q} \cap (-n, n)$  with  $0 < t - s < \delta$  and  $(s, t] \cap \{u_1, \dots, u_m\} = \emptyset$ .

Note that the inequalities between functions required in (17) hold whenever the same inequalities hold between their restrictions to  $\mathbb{Q}$ , by left and right continuity. Note also that condition (18) is equivalent to the following condition involving quantifiers only over countable sets:

for all rationals  $\varepsilon > 0$  and all  $n \in \mathbb{N}$ , there exist a rational  $\delta > 0$  and an  $m \in \mathbb{Z}^+$  such that, for all rationals  $\eta > 0$ , there exist rationals  $s_1, t_1, \dots, s_m, t_m \in (-n, n)$ , with  $s_i < t_i$  for all  $i$  and with  $\sum_{i=1}^m (t_i - s_i) < \eta$ , such that

$$\|\Phi_{(s,t]} - \text{id}\| < \varepsilon$$

whenever  $s, t \in \mathbb{Q} \cap (-n, n)$  with  $0 < t - s < \delta$  and  $(s, t] \cap ((s_1, t_1] \cup \dots \cup (s_m, t_m]) = \emptyset$ .

Denote by  $D_E^{*,+}$  the set of those  $z \in D_E$  where conditions (15), (16), (17) and (18) all hold. Then  $D_E^{*,+}$  is a measurable subset of  $D_E$ . Fix  $z \in D_E^{*,+}$ . Given a bounded interval  $I$ , we can find sequences of rationals  $s_n$  and  $t_n$  such that  $(s_n, t_n] \rightarrow I$  as  $n \rightarrow \infty$ . Then, by conditions (17) and (18),

$$d_{\mathcal{D}}(\Phi_{(s_n, t_n]}, \Phi_{(s_m, t_m]}) \leq \|\Phi_{(s_n, s_m]} - \text{id}\| + \|\Phi_{(t_n, t_m]} - \text{id}\| \rightarrow 0$$

as  $n, m \rightarrow \infty$ . So the sequence  $\Phi_{(s_n, t_n]}$  converges in  $\mathcal{D}$ , with limit  $\Phi_I$ , say, and  $\Phi_I$  does not depend on the approximating sequences of rationals. In the case where  $I = I_1 \oplus I_2$ , there exists another sequence of rationals  $u_n$  such that  $(s_n, u_n] \rightarrow I_1$  and  $(u_n, t_n] \rightarrow I_2$  as  $n \rightarrow \infty$ .

Hence  $\Phi = (\Phi_I : I \subseteq \mathbb{R})$  has the weak flow property, by Proposition 8.3. It is straightforward to deduce from (18) that  $\Phi$  is moreover cadlag, so  $\Phi = \Phi(z) \in D^\circ(\mathbb{R}, \mathcal{D})$ . It follows from its construction, and the preceding proposition, that the map  $z \mapsto \Phi(z) : D_E^{*,+} \rightarrow D^\circ(\mathbb{R}, \mathcal{D})$  is measurable.

Now, for all  $z \in D_E^{*,+}$ , we have  $Z^{E,+}(\Phi(z)) = z$  and for all  $\phi \in D^\circ(\mathbb{R}, \mathcal{D})$ , we have  $Z^{E,+}(\phi) \in D_E^{*,+}$  and  $\Phi(Z^{E,+}(\phi)) = \phi$ . Hence  $D_E^{\circ,+} = D_E^{*,+}$  and  $Z^{E,+} : D^\circ(\mathbb{R}, \mathcal{D}) \rightarrow D_E^{\circ,+}$  is a bijection with inverse  $\Phi^{E,+} = \Phi$ .

Consider now for  $z \in D_E$  the condition

$$z_t^{(s,x)} = \sup_{y \in \mathbb{Q}, y < x} z_t^{(s,y)}, \quad (s, x) \in E, \quad t \in \mathbb{Q}, \quad t > s. \quad (19)$$

Denote by  $D_E^{*, -}$  the set of those  $z \in D_E$  where conditions (15), (17), (18) and (19) all hold, and define  $\Phi$  on  $D_E^{*, -}$  exactly as on  $D_E^{*, +}$ . Then, by a similar argument,  $D_E^{\circ, -} = D_E^{*, -}$  and  $Z^{E, -} : D^\circ(\mathbb{R}, \mathcal{D}) \rightarrow D_E^{\circ, -}$  is a bijection with inverse  $\Phi^{E, -} = \Phi$ . In particular  $\Phi^{E, +} = \Phi^{E, -}$  on  $D_E^{\circ, -} \cap D_E^{\circ, +}$  and so  $D_E^\circ = D_E^{\circ, -} \cap D_E^{\circ, +}$ , as claimed.  $\square$

**Proposition 8.10.** *Let  $E$  be a countable subset of  $\mathbb{R}^2$  containing  $\mathbb{Q}^2$ . Then  $\mu_E(C_E^\circ) = 1$ .*

*Proof.* We use an identification of  $C_E^\circ$  analogous to that implied for  $D_E^\circ$  by the preceding proof. The same five conditions (15), (16), (17), (18) and (19) characterize  $C_E^\circ$  inside  $C_E$ , except that, in (18), only the case  $m = 0$  is allowed. Recall that, under  $\mu_E$ , for time-space starting points  $e = (s, x)$  and  $e' = (s', x')$ , the coordinate processes  $Z^e$  and  $Z^{e'}$  behave as independent Brownian motions up to

$$T^{ee'} = \inf\{t \geq s \vee s' : Z_t^e - Z_t^{e'} \in \mathbb{Z}\},$$

after which they continue to move as Brownian motions, but now with a constant separation. In particular, if  $s = s'$  and  $x' = x + n$  for some  $n \in \mathbb{Z}$ , then  $T^{ee'} = 0$ , so  $Z_t^{e'} = Z_t^e + n$  for all  $t \geq s$ , so (15) holds almost surely.

Let  $(s, x) \in E$  and  $t, u \in \mathbb{Q}$ , with  $s \leq t < u$ . Consider the event

$$A = \left\{ \sup_{y \in \mathbb{Q}, y < Z_t^{(s,x)}} Z_u^{(t,y)} = Z_u^{(s,x)} = \inf_{y' \in \mathbb{Q}, y' > Z_t^{(s,x)}} Z_u^{(t,y')} \right\}.$$

Fix  $n \in \mathbb{N}$  and set  $Y = n^{-1} \lfloor n Z_t^{(s,x)} \rfloor$  and  $Y' = Y + 1/n$ . Then  $Y$  and  $Y'$  are  $\mathcal{F}_t$ -measurable,  $\mathbb{Q}$ -valued random variables. Now  $\mathbb{P}(Y < Z_t^{(s,x)} < Y') = 1$  and

$$\{Y < Z_t^{(s,x)} < Y'\} \cap \{T^{(t,Y)(t,Y')} \leq u\} \subseteq A.$$



By the Markov property of Brownian motion, almost surely,

$$\mathbb{P}(T^{(t,Y)}(t,Y') \leq u | \mathcal{F}_t) \geq 2\Phi\left(\frac{1}{n\sqrt{2(u-t)}}\right),$$

and the right-hand side tends to 1 as  $n \rightarrow \infty$ . So, by bounded convergence, we obtain  $\mathbb{P}(A) = 1$ . On taking a countable intersection of such sets  $A$  over the possible values of  $s, x, t$  and  $u$ , we deduce that conditions (16), (17) and (19) hold almost surely.

It remains to establish the continuity condition (18). For a standard Brownian motion  $B$  starting from 0, we have, for  $n \geq 4$ ,

$$\mathbb{P}\left(\sup_{t \leq 1} |B_t| > n\right) \leq e^{-n^2/2}.$$

Define, for  $\delta > 0$  and  $e = (s, x) \in E$ ,

$$V^e(\delta) = \sup_{s \leq t \leq s+\delta^2} |Z_t^e - x|.$$

Then, by scaling,

$$\mathbb{P}(V^e(\delta) > n\delta) \leq e^{-n^2/2}.$$

Consider, for each  $n \in \mathbb{N}$  the set

$$E_n = \{(j2^{-2n}, k2^{-n}) : j \in \frac{1}{2}\mathbb{Z} \cap [-2^{2n}, 2^{2n}), k = 0, 1, \dots, 2^n - 1\}$$

and the event

$$A_n = \bigcup_{e \in E_n} \{V^e(2^{-n}) > n2^{-n}\}.$$

Then  $\mathbb{P}(A_n) \leq |E_n|e^{-n^2/2}$ , so  $\sum_n \mathbb{P}(A_n) < \infty$ , so by Borel–Cantelli, almost surely, there is a random  $N < \infty$  such that  $V^e(2^{-n}) \leq n2^{-n}$  for all  $e \in E_n$ , for all  $n \geq N$ .

Given  $\varepsilon > 0$ , choose  $n \geq N$  such that  $(4n+2)2^{-n} \leq \varepsilon$  and set  $\delta = 2^{-2n-1}$ . Then, for all rationals  $s, t \in (-n, n)$  with  $0 < t - s < \delta$  and all rationals  $x \in [0, 1]$ , there exist  $e^\pm = (r, y^\pm) \in E_n$  such that

$$r \leq s < t \leq r + 2^{-2n}, \quad x + n2^{-n} < y^+ \leq x + (n+1)2^{-n}, \quad x - (n+1)2^{-n} \leq y^- < x - n2^{-n},$$

Then,  $Z_s^{e^-} < x < Z_s^{e^+}$ , so

$$x - \varepsilon \leq Z_t^{e^-} \leq Z_t^{(s,x)} \leq Z_t^{e^+} \leq x + \varepsilon.$$

Hence  $\|\Phi_{(s,t]} - \text{id}\| \leq \varepsilon$ , as required.  $\square$

Recall that  $\Phi^E$  denotes the common restriction of  $\Phi^{E,+}$  and  $\Phi^{E,-}$  to  $D_E^\circ$ .

**Proposition 8.11.** *Let  $E = \mathbb{Q}^2$ . Then  $\Phi^E$  is continuous at  $z$  for all  $z \in C_E^\circ$ .*

*Proof.* Consider a sequence  $(z_k : k \in \mathbb{N})$  in  $D_E^\circ$  and suppose that  $z_k \rightarrow z$  in  $D_E$ , with  $z \in C_E^\circ$ . Set  $\phi^k = \Phi^E(z_k)$  and  $\phi = \Phi^E(z)$ . It will suffice to show that, for all  $n \in \mathbb{N}$ , we have  $d_C^{(n)}(\phi^k, \phi) \rightarrow 0$  as  $k \rightarrow \infty$ . Given  $\varepsilon > 0$ , choose  $\varepsilon' > 0$  and  $\eta > 0$  so that  $\varepsilon' + 2\eta < \varepsilon$ . Then choose  $m \in \mathbb{N}$  so that  $\varepsilon' + 2\eta + 1/m < \varepsilon$  and so that  $\|\phi_{ts} - \text{id}\| < \eta$  for all  $s, t \in (-n, n)$  with  $0 < t - s < 1/m$ . Consider the finite set

$$F = (m^{-1}\mathbb{Z} \cap [-n, n)) \times (m^{-1}\mathbb{Z} \cap [0, 1)).$$

There exists a  $K < \infty$  such that, for all  $k \geq K$ , all  $(s_0, x_0) \in F$ , and all  $t \in (s_0, n]$ ,

$$|\phi_{(s_0, t]}^{k,+}(x_0) - \phi_{ts_0}^+(x_0)| = |\phi_{(s_0, t]}^{k,-}(x_0) - \phi_{ts_0}^-(x_0)| < \varepsilon'.$$

For all  $s \in [-n, n)$  and all  $x \in [0, 1)$ , there exists  $(s_0, x_0) \in F$  such that

$$s_0 \leq s < s_0 + 1/m, \quad x_0 \leq x + \varepsilon' + \eta + 1/m < x_0 + 1/m.$$

Then

$$\phi_{(s_0, s]}^{k,+}(x_0) \geq \phi_{ss_0}^+(x_0) - \varepsilon' \geq x_0 - \varepsilon' - \eta > x,$$

so

$$\phi_{(s_0, t]}^{k,+}(x_0) \geq \phi_{(s, t]}^{k,+}(x), \quad t \geq s.$$

Also, we have

$$\phi_{ss_0}^+(x_0) \leq x_0 + \eta \leq x + \varepsilon' + 2\eta + 1/m < x + \varepsilon,$$

so

$$\phi_{ts_0}^+(x_0) \leq \phi_{ts}^+(x + \varepsilon), \quad t \geq s.$$

Now, for all  $t \in (s, n]$ ,

$$\phi_{(s_0, t]}^{k,+}(x_0) \leq \phi_{ts_0}^+(x_0) + \varepsilon,$$

so

$$\phi_{(s, t]}^{k,+}(x) \leq \phi_{ts}^+(x + \varepsilon) + \varepsilon.$$

By a similar argument, for all  $t \in (s, n]$ ,

$$\phi_{(s, t]}^{k,-}(x) \geq \phi_{ts}^-(x - \varepsilon) - \varepsilon,$$

so  $d_{\mathcal{D}}(\phi_{(s, t]}^k, \phi_{(s, t]}) \leq \varepsilon$ . Hence  $d_C^{(n)}(\phi^k, \phi) \rightarrow 0$  as  $k \rightarrow \infty$ , as required.  $\square$

### 8.3 From the coalescing Brownian flow to the Brownian webs

The coalescing Brownian flow, which provides the limit object for our main result is a refinement, in certain respects, of Arratia's flow of coalescing Brownian motions, via the work of Tóth and Werner. In a series of works, beginning with [6], Fontes, Newman and others have already provided another such refinement, in fact several, which they call Brownian webs. The direction taken by Fontes et al. emphasises path properties: the Brownian web is conceived as a random element of a space  $\mathcal{H}$  of compact collections of  $\mathbb{R}$ -valued paths with specified starting points. In our formulation, one does not see so clearly the possible varieties of path, but we find the state space  $C^0(\mathbb{R}, \bar{\mathcal{D}})$  a convenient one, with a natural time-reversal map, and just one candidate for a probability measure corresponding to Arratia's flow. We now attempt to clarify the relationship between the coalescing Brownian flow and one of the Brownian webs.

In [6], the authors make the comment that there is more than one natural distribution of an  $\mathcal{H}$ -valued<sup>8</sup> random variable  $\mathcal{W}$  that satisfies the following two conditions.

- (i) From any deterministic point  $(x, t)$  in space-time, there is almost surely a unique path  $W_{x,t}$  in  $\mathcal{W}$  starting from  $(x, t)$ .
- (ii) For any deterministic  $n$  and  $(x_1, t_1), \dots, (x_n, t_n)$ , the joint distribution of  $W_{x_1, t_1}, \dots, W_{x_n, t_n}$  is that of coalescing Brownian motions (with unit diffusion constant).

The *standard Brownian web* satisfies (i) and (ii) together with a certain minimality condition. On the other hand, the *forward full Brownian web*, introduced in [7], satisfies (i) and (ii) together with a certain maximality condition, subject to a non-crossing condition. Also in [7], the authors characterize a third object, the *full Brownian web*, which is a random variable on the space  $\mathcal{H}^F$  of compact collections of paths from  $\mathbb{R} \rightarrow \mathbb{R}$ , and explain how this is naturally related to the other Brownian webs.

We now describe in detail the space  $\mathcal{H}^F$  of the full Brownian web and give its characterization. We then show that there is a natural way to realise the full Brownian web (and hence also the standard Brownian web and forward full Brownian web) as a random variable on  $C^0(\mathbb{R}, \mathcal{D})$ . Define the function  $\Phi : [-\infty, \infty] \times \mathbb{R} \rightarrow [0, 1]$  by

$$\Phi(x, t) = \frac{\tanh(x)}{1 + |t|}.$$

Now construct the two metric spaces  $(\Pi^F, d)$  and  $(\mathcal{H}^F, d_{\mathcal{H}^F})$  as follows. Let  $\Pi^F$  denote the set of functions  $f : \mathbb{R} \rightarrow [-\infty, \infty]$  such that  $\Phi(f(t), t)$  is continuous, and define

$$d^F(f_1, f_2) = \sup_{t \in \mathbb{R}} |\Phi(f_1(t), t) - \Phi(f_2(t), t)|.$$

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<sup>8</sup>We refer to [6] for precise definitions and attempt here only to give a flavour of their results.

Then  $(\Pi^F, d^F)$  is a complete separable metric space. Now let  $\mathcal{H}^F$  denote the set of compact subsets of  $(\Pi^F, d^F)$ , with  $d_{\mathcal{H}^F}$  the induced Hausdorff metric

$$d_{\mathcal{H}^F}(K_1, K_2) = \sup_{g_1 \in K_1} \inf_{g_2 \in K_2} d^F(g_1, g_2) \vee \sup_{g_2 \in K_2} \inf_{g_1 \in K_1} d^F(g_1, g_2).$$

The space  $(\mathcal{H}^F, d_{\mathcal{H}^F})$  is also complete and separable. Let  $\mathcal{F}_{\mathcal{H}^F}$  be the Borel  $\sigma$ -algebra on  $\mathcal{H}^F$ . The full Brownian web is defined in [7] as follows.

**Definition 8.12.** *A full Brownian web  $\bar{W}^F$  is any  $(\mathcal{H}^F, \mathcal{F}_{\mathcal{H}^F})$ -valued random variable whose distribution has the following properties.*

- (a) *Almost surely the paths of  $\bar{W}^F$  are noncrossing (although they may touch, including coalescing and bifurcating).*
- (b<sub>1</sub>) *From any deterministic point  $(x, t) \in \mathbb{R}^2$ , there is almost surely a unique path  $W_{x,t}^F$  passing through  $x$  at time  $t$ .*
- (b<sub>2</sub>) *For any deterministic  $m$ ,  $\{(x_1, t_1), \dots, (x_m, t_m)\}$ , the joint distribution of the semipaths  $\{W_{x_j, t_j}^F(t), t \geq t_j, j = 1, \dots, m\}$  is that of a flow of coalescing Brownian motions (with unit diffusion constant).*

The authors show that any two full Brownian webs have the same distribution.

Fix  $E$ , a countable subset of  $\mathbb{R}^2$  containing  $\mathbb{Q}^2$ . We define a map  $\theta_F : C^\circ(\mathbb{R}, \bar{\mathcal{D}}) \rightarrow \mathcal{H}^F$  as follows. Let

$$\Pi = \bigcup_{t \in \mathbb{R}} C([t, \infty), [-\infty, \infty]).$$

Let  $\bar{\mathcal{H}}$  be the set of all subsets  $A$  of  $\Pi \cup \Pi^F$  having the the following noncrossing property

$$\text{for all } f, g \in A \text{ and all } s, t \in \text{dom}(f) \cap \text{dom}(g), f(s) < g(s) \text{ implies } f(t) \leq g(t).$$

We say that a set  $U \in \bar{\mathcal{H}}$  is *maximal* if, for any  $f \in \Pi \cup \Pi^F$ ,  $U \cup \{f\} \in \bar{\mathcal{H}}$  implies  $f \in U$ . For  $\phi \in C^\circ(\mathbb{R}, \bar{\mathcal{D}})$ , define  $\theta_0(\phi) = \{Z^{e,+}(\phi) : e \in E\}$ . Note that, since  $E$  is dense in  $\mathbb{R}^2$ , if  $f, g \in \Pi \cup \Pi^F$  and  $\theta_0(\phi) \cup \{f\}, \theta_0(\phi) \cup \{g\} \in \bar{\mathcal{H}}$ , then also  $\theta_0(\phi) \cup \{f, g\} \in \bar{\mathcal{H}}$ . Hence there exists a unique maximal set  $\theta_{FF}(\phi) \in \bar{\mathcal{H}}$  containing  $\theta_0(\phi)$ . Furthermore, this set is independent of the choice of  $E$ . Define  $\theta_F(\phi) = \theta_{FF}(\phi) \cap \Pi^F$ .

This can be viewed as a random variable on  $C^\circ(\mathbb{R}, \bar{\mathcal{D}})$  as a consequence of Propositions 8.13 and 8.15 below. Note that, under the measure  $\mu_W$ , the conditions to be a full Brownian web are almost surely satisfied: (a) is immediate from the construction; (b<sub>1</sub>) follows from Proposition 8.14, where condition (i) holds almost surely by Proposition 8.10, and condition (ii) depends on a countable number of almost sure conditions; (b<sub>2</sub>) follows from Theorem 3.2. In fact  $\theta_{FF}$  defined above is a forward full Brownian web, and the standard Brownian web can be constructed similarly, but we omit the details here.

**Proposition 8.13.** *The set  $\theta_F(\phi) \subset \Pi^F$  is compact.*

*Proof.* Suppose that  $f_1, f_2, \dots \in \theta_F(\phi)$ . Since paths in  $\theta_F(\phi)$  are noncrossing, there exists a subsequence  $n_r$  such that  $f_{n_r}(t)$  is monotone for all  $t \in \mathbb{R}$  and so there exists some  $f : \mathbb{R} \rightarrow [-\infty, \infty]$  such that  $f_{n_r} \rightarrow f$  pointwise.

Given  $\varepsilon > 0$ , there exist  $n > 2\varepsilon^{-1}$ , and  $-n = a_0 < \dots < a_M = n$  such that  $\|\phi_{sa_k}^+ - \text{id}\| < \frac{\varepsilon}{4}$  for all  $a_k \leq s \leq a_{k+1}$ ,  $k = 0, \dots, M-1$ . Pick  $N$  sufficiently large that if  $n_r > N$ , for  $k = 1, \dots, M-1$ ,  $|f_{n_r}(a_k) - f(a_k)| < \frac{\varepsilon}{4}$  if  $|f(a_k)| < \infty$  and  $|\tanh(f_{n_r}(a_k)) - \tanh(f(a_k))| < \frac{\varepsilon}{4}$  if  $|f(a_k)| = \infty$ . Let  $a_k \leq s < a_{k+1}$ . First suppose that  $|f(a_k)| < \infty$ . Then

$$|\tanh(f_{n_r}(s)) - \tanh(f(s))| \leq |f_{n_r}(s) - f(s)| \leq |\phi_{sa_k}^+(f(a_k) + \frac{\varepsilon}{4}) - \phi_{sa_k}^+(f(a_k) - \frac{\varepsilon}{4})| < \varepsilon.$$

Now suppose  $f(a_k) = \infty$ . Then

$$|\tanh(f_{n_r}(a_k)) - 1| < \frac{\varepsilon}{4}$$

and so

$$f_{n_r}(a_k) > \tanh^{-1}(1 - \frac{\varepsilon}{4}).$$

Therefore

$$f_{n_r}(s) \geq \phi_{sa_k}^+(\tanh^{-1}(1 - \frac{\varepsilon}{4})) > \tanh^{-1}(1 - \frac{\varepsilon}{4}) - \frac{\varepsilon}{4},$$

and so

$$f(s) \geq \tanh^{-1}(1 - \frac{\varepsilon}{4}) - \frac{\varepsilon}{4}.$$

Hence  $f(s) = \infty$  and  $\tanh(f_{n_r}(s)) \in (1 - \frac{\varepsilon}{2}, 1]$ , and so  $|\tanh(f_{n_r}(s)) - \tanh(f(s))| < \varepsilon$ . A similar argument holds when  $f(a_k) = -\infty$ . Therefore,

$$d^F(f_{n_r}, f) \leq \sup_{-n < t < n} |\tanh(f_{n_r}(t)) - \tanh(f(t))| \vee 2/(n+1) \leq \varepsilon$$

and so  $f_{n_r} \rightarrow f$  in  $(\Pi^F, d^F)$ . Therefore,  $f$  is continuous and  $\theta_{FF}(\phi) \cup \{f\}$  is noncrossing, and so  $f \in \theta_F(\phi)$  proving compactness.  $\square$

**Proposition 8.14.** *For every  $(x, s) \in \mathbb{R}^2$ , there exists some  $f \in \theta_F(\phi)$  with  $f(s) = x$ . Moreover,  $f$  is unique if the following two conditions hold.*

- (i)  $\phi_{ts}^+(x) = \phi_{ts}^-(x)$  for all  $t \geq s$ ;
- (ii)  $\phi_{st}^+(y) \neq x$  for all  $(y, t) \in \mathbb{Q}^2$  with  $t < s$ .

*Proof.* Define a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  by

$$f(t) = \begin{cases} \inf\{y : \phi_{st}^+(y) > x\} & \text{if } t < s \\ \phi_{ts}^+(x) & \text{if } t \geq s. \end{cases}$$

We first show that  $f$  is continuous. Clearly this is the case on  $[s, \infty)$ . Suppose  $t < u \leq s$ . Pick a sequence  $y_n \downarrow f(t)$ . By the weak flow property of  $\phi$ ,  $\phi_{su}^+(\phi_{ut}^+(y_n)) \geq \phi_{st}^+(y_n) > x$  and so  $\phi_{ut}^+(y_n) \geq f(u)$  for all  $n$ . Letting  $n \rightarrow \infty$  gives  $\phi_{ut}^+(f(t)) \geq f(u)$ . Similarly, using  $\inf\{y : \phi_{st}^+(y) > x\} = \sup\{y : \phi_{st}^-(y) \leq x\}$ ,  $\phi_{ut}^-(f(t)) \leq f(u)$ . Now given  $\varepsilon > 0$ , there exists some  $\delta > 0$  such that  $\|\phi_{ut} - \text{id}\| < \varepsilon$  for all  $0 < u - t < \delta$ . But then  $|f(t) - f(u)| \leq |f(t) - \phi_{ut}^-(f(t))| \wedge |f(t) - \phi_{ut}^+(f(t))| < \varepsilon$ , proving continuity. Also, by construction,  $\theta_0(\phi) \cup \{f\} \in \bar{\mathcal{H}}$  and so  $f \in \theta_F(\phi)$  as required.

For uniqueness, suppose  $f_1, f_2 \in \theta_F(\phi)$  with  $f_1(s) = f_2(s) = x$ . By the noncrossing property, for all  $t \geq s$ ,  $\phi_{ts}^-(x) \leq f_i(t) \leq \phi_{ts}^+(x)$ . Condition (i) therefore implies  $f_1(t) = f_2(t)$  for all  $t \geq s$ . If  $t < s$ , then  $\sup\{y : \phi_{st}^-(y) < x\} \leq f_i(t) \leq \inf\{y : \phi_{st}^+(y) > x\}$ . If  $\inf\{y : \phi_{st}^+(y) > x\} - \sup\{y : \phi_{st}^-(y) < x\} > 0$ , then by continuity, there exist rationals  $(y, u)$  with  $\phi_{st}^+(y) = x$ . Hence (ii) ensures  $f_1(t) = f_2(t)$  for all  $t < s$ .  $\square$

**Proposition 8.15.** *The function  $\theta_F : C^\circ(\mathbb{R}, \bar{\mathcal{D}}) \rightarrow \mathcal{H}^F$  is continuous.*

*Proof.* A sequence  $\phi^m \rightarrow \phi$  in  $(C^\circ(\mathbb{R}, \bar{\mathcal{D}}), d_C)$  if and only if  $d_C^{(n)}(\phi^m, \phi) \rightarrow 0$  for all  $n \in \mathbb{N}$ . Since  $d^F(f, g) \leq d^{(n)}(f, g) \vee 2/(n+1)$ , where

$$d^{(n)}(f, g) = \sup_{-n < t < n} |f(t) - g(t)|,$$

in order to show that  $\theta_F$  is continuous, it is enough to show that for every  $n$ , the distance  $d_{\mathcal{H}^F}^{(n)}(\theta_F(\phi^1), \theta_F(\phi^2)) \leq d_C^{(n)}(\phi^1, \phi^2)$  for all  $\phi^1, \phi^2 \in C^\circ(\mathbb{R}, \bar{\mathcal{D}})$ , where

$$d_{\mathcal{H}^F}^{(n)}(K_1, K_2) = \sup_{g_1 \in K_1} \inf_{g_2 \in K_2} d^{(n)}(g_1, g_2) \vee \sup_{g_2 \in K_2} \inf_{g_1 \in K_1} d^{(n)}(g_1, g_2).$$

Suppose  $\phi^1, \phi^2 \in C^\circ(\mathbb{R}, \bar{\mathcal{D}})$ . By compactness, there exist  $f_1 \in \theta_F(\phi^1)$ ,  $f_2 \in \theta_F(\phi^2)$  such that  $d^{(n)}(f_1, f_2) = d_{\mathcal{H}^F}^{(n)}(\theta_F(\phi^1), \theta_F(\phi^2))$ . Without loss of generality, suppose that  $f_2$  is chosen so that  $d^{(n)}(f_1, f_2) = \inf_{g \in \theta_F(\phi^2)} d^{(n)}(f_1, g) = f_2(s) - f_1(s)$  for some  $s \in \mathbb{R}$ . Let  $f_1(s) = x$  and  $f_2(s) = y$ . By continuity, there exists some  $t > s$  such that  $f_1(t) - \phi_{ts}^-(y) = d^{(n)}(f_1, f_2)$ , otherwise it would be possible to pick some  $g \in \theta_F(\phi^2)$  with  $d^{(n)}(f_1, g) < d^{(n)}(f_1, f_2)$ . Since  $f_1(t) \in [\phi_{ts}^{1,-}(x), \phi_{ts}^{1,+}(x)]$ ,

$$\phi_{ts}^{2,-}(y) - \phi_{ts}^{1,+}(x) \leq x - y \leq \phi_{ts}^{2,+}(y) - \phi_{ts}^{1,-}(x).$$

Hence there exists some

$$u \in \left[ \frac{1}{2}(x + \phi_{ts}^{1,-}(x)), \frac{1}{2}(x + \phi_{ts}^{1,+}(x)) \right] \cap \left[ \frac{1}{2}(y + \phi_{ts}^{2,-}(y)), \frac{1}{2}(y + \phi_{ts}^{2,+}(y)) \right].$$

Therefore,

$$\begin{aligned}
d_{\mathcal{H}^F}(\theta_F(\phi^1), \theta_F(\phi^2)) &= |x - y| \\
&= |\phi_{ts}^{1\times}(u) - \phi_{ts}^{2\times}(u)| \\
&\leq \|\phi_{ts}^{1\times} - \phi_{ts}^{2\times}\| \\
&\leq d_C^{(n)}(\phi^1, \phi^2),
\end{aligned}$$

as required. □

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